ON TERNARY CUBIC FORMS

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Abstract. For ternary cubic forms $f_a: \ u^3+v^3+w^3+3auvw$ a new approach is pursued to estimate their minimum in the sense of the Geometry of numbers. The idea is to inscribe into the star body $|f_a| \le 1$ a suitably rotated and dilated copy of the double paraboloid $x^2+y^2+|z| \le 1$ whose critical determinant has been recently evaluated by the author [14]. For $-2.31788 < a < -0.48403$, $a \ne -1$, the result obtained is the best of its kind known so far.

1. Introduction. Survey of classic results. Let $\mathfrak{f} = \mathfrak{f}(u,v,w)$ denote a cubic form (i.e., a homogeneous polynomial of degree 3) with real coefficients. We shall suppose throughout that $\mathfrak{f}$ is *regular*, i.e., that $\frac{\partial \mathfrak{f}}{\partial u}, \frac{\partial \mathfrak{f}}{\partial v}, \frac{\partial \mathfrak{f}}{\partial w}$ vanish simultaneously only at the origin[1]. The following number theoretic question is natural: How small can $|\mathfrak{f}|$ be made by a suitable choice of *integer* values u,v,w (not all zero) - with the idea that the desired answer should provide a certain amount of uniformity in the coefficients of $\mathfrak{f}$.

It is known (cf. Weber [17], p. 401 - 408[2]) that – for $\mathfrak{f}$ regular – there always exists a non-singular real linear transformation which reduces $\mathfrak{f}$ to the canonical form

$$f_a: \quad u^3+v^3+w^3+3a\,uvw \qquad (a \ne -1)\,. \tag{1.1}$$

Of course $\mathbb{Z}^3$ is then transformed into a general three-dimensional lattice Λ. Therefore, it was natural to define

$$M_a = \sup_{\Lambda:\ d(\Lambda)=1} \ \inf_{\substack{(u,v,w)\in\Lambda \\ (u,v,w)\ne(0,0,0)}} \left|u^3+v^3+w^3+3a\,uvw\right| \,, \tag{1.2}$$

[1] We shall say more about this condition (and also about the forms *not* meeting it) in the Appendix.

[2] The author is indebted to Professor Kurt Girstmair (Innsbruck) for this important reference.

with Λ ranging over all three-dimensional lattices with lattice constant[3] $d(\Lambda) = 1$. This was simply called the *minimum* of the ternary cubic form involved.

We briefly report about two special cases excluded in (1.1). (Here $(\frac{\partial \mathfrak{f}}{\partial u}, \frac{\partial \mathfrak{f}}{\partial v}, \frac{\partial \mathfrak{f}}{\partial w}) = (0,0,0)$ has nontrivial solutions.) The case $\mathfrak{f}:\ uvw$ is known as the problem of *the product of three* (real) *linear forms*: This was successfully attacked by Davenport [1], [2], [3], [4], [5], who ultimately proved that the minimum (in the same sense as (1.2)) of this form is equal to $\frac{1}{7}$. For $a = -1$, f_{-1} is a product of three linear forms L_1, L_2, L_3, with L_1 real and L_3 the complex conjugate of L_2. This was also dealt with by Davenport [3], with the result that

$$M_{-1} = \sqrt{\frac{27}{23}} = 1.08347\ldots\,. \tag{1.3}$$

For general a, only more or less precise bounds for M_a are known. The problem is connected with the notion of the *critical determinant*[4] $\Delta(K_a)$ of the star body $K_a:\ |f_a| \le 1$. By a simple homogeneity consideration,

$$M_a = \frac{1}{\Delta(K_a)}\,. \tag{1.4}$$

Special attention was paid to the case $a = 0$: Davenport [6] inscribed into K_0 the convex body

$$u_+^3 + v_+^3 + w_+^3 \le 1\,, \quad (-u)_+^3 + (-v)_+^3 + (-w)_+^3 \le 1\,, \qquad \text{with } z_+ := \max(z,0)\,,$$

and applied Minkowski's Convex Body Theorem. He thus obtained

$$M_0 \le 8\Gamma^{-3}\left(\frac{4}{3}\right)\left(2 + \frac{27\sqrt{3}}{2\pi}\right)^{-1} = 1.1897\ldots\,. \tag{1.5}$$

Later [7], he used the more subtle non-convex body

$$\theta(u^3) + \theta(v^3) + \theta(w^3) \le 1\,,\ \theta(-u^3) + \theta(-v^3) + \theta(-w^3) \le 1\,, \quad \theta(z) := \begin{cases} z & \text{for } z \ge 0, \\ \frac{z}{9} & \text{for } z < 0, \end{cases}$$

and applied Blichfeldt's Theorem to conclude that

$$M_0 \le 8\Gamma^{-3}\left(\frac{4}{3}\right)\left(2 + \frac{27\sqrt{3}}{2\pi}\left(1 + \frac{2}{3}\sum_{n=1}^{\infty} 9^{-n}\left(\frac{1}{3n+1} + \frac{1}{3n+2}\right)\right)\right)^{-1} = 1.1571\ldots\,. \tag{1.6}$$

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3) For the basic concepts of the Geometry of Numbers the reader may consult, e.g., the enlightening textbook by Gruber and Lekkerkerker [10]. There also a survey of the literature on forms of other degrees or another number of variables can be found.

4) This is the infimum of the lattice constants of all lattices of which no nontrivial point is contained in the interior of the star body.

For arbitrary a, an obvious possibility to estimate $\Delta(K_a)$ and thus M_a is to inscribe into K_a a convex body of the shape $|u|^3+|v|^3+|w|^3\le c$ and to apply Minkowski's Convex Body Theorem. Thus one gets

$$M_a \le \frac{1}{\Gamma^3\left(\frac{4}{3}\right)}(1+|a|)\,.$$

MORDELL [12] applied a deep method involving the concept of a *polar reciprocal lattice* and the reduction of the problem to a two-dimensional one, to establish better upper bounds for all a. It was noted by GOLSER (a student of E. HLAWKA) that these estimates could be improved substantially, by a refinement of Mordell's own method. His final results may be stated as follows: Writing $k=3a$, let

$$\mu(k)=\begin{cases}\max\left(\frac{1}{27}(k^4+108\,k)+\frac{1}{2}k^3+2k^2+3,\ \frac{1}{2}k^3+2k^2+27\right) & \text{for } k\ge 0,\\ \max\left(\frac{1}{27}(-k^4+108\,|k|)+\frac{1}{2}|k|^3+2k^2+3,\ \frac{1}{2}|k|^3+2k^2+27\right) & \text{for } -\sqrt[3]{108}<k<0,\\ \max\left(\frac{1}{27}(k^4-108\,|k|)+\frac{1}{2}|k|^3+2k^2+27,\ \frac{1}{2}|k|^3+2k^2+3\right) & \text{for } k\le -\sqrt[3]{108}.\end{cases}$$

Then for all a (GOLSER [8])

$$M_a \le \left(\frac{2\mu(3a)}{23}\right)^{1/4}\,. \tag{1.7}$$

GOLSER [8], [9] also noted that, for certain ranges of a, better bounds can be obtained by the simple procedure to inscribe a sphere into K_a.

Later on, the author [13] refined this idea by using a more general ellipsoid of the shape

$$u^2+v^2+w^2+2t(uv+uw+vw)\le r^2\,,$$

with a parameter $t\in\,]-\frac{1}{2},1[,\ t\ne 0$. This leads to the result

$$M_a \le \sqrt{2}\,(1-t)\sqrt{1+2t}\,\max\left(\frac{|1+a|}{\sqrt{3}\,(1+2t)^{3/2}},\ \phi_1(t),\ \phi_2(t)\right)\,, \tag{1.8}$$

where, for $j=1,2$,

$$\phi_j(t) := \left(2+2t+4c_j(t)\,t+c_j(t)^2\right)^{-3/2}\left|2+3ac_j(t)\,+c_j(t)^3\right|\,,$$

$$c_j(t) := \frac{a-1-2t+(-1)^j\sqrt{(a-1)^2+4t+4(a-1)t^2}}{2t}\,,$$

$\phi_j(t):=0$ if $c_j(t)\notin\mathbb{R}$. For any given a, the parameter t can be chosen to make the estimate optimal.

The novelty of the present paper is based on the author's recent result [14] that the critical determinant of the *double paraboloid*

$$\mathcal{P}:\quad |z|+x^2+y^2\le 1$$

is given by

$$\Delta(\mathcal{P}) = \frac{1}{2}\,. \tag{1.9}$$

Inscribing a suitably rotated and dilated copy of $\mathcal{P}$ into K_a, we shall infer an estimate for $\Delta(K_a)$, resp., M_a, which for a certain range of a (namely $-2.31788 < a < -0.48403$, $a \neq -1$) improves upon all bounds known so far.

Before entering into the details of this new appoach, it might be worthwhile to provide a table[5)] which compares the efficiency of the different methods mentioned above and to indicate which of them "holds the record" for a certain value of the constant a.

Range for a	Best bound for M_a
$a < -6.649$	(1.7), Golser [8]
$-6.649 < a < -2.318$	(1.8), Nowak [13]
$-2.318 < a < -0.484$, $a \neq -1$	present paper
$a = -1$	(1.3), Davenport [3]
$-0.484 < a < -0.02685$	(1.7), Golser [8]
$-0.02685 < a < 0.0407$, $a \neq 0$	Davenport [6]
$a = 0$	(1.6), Davenport [7]
$0.0407 < a < 0.819$	(1.7), Golser [8]
$0.819 < a < 6.76$	(1.8), Nowak [13]
$a > 6.76$	(1.7), Golser [8]

2. The paraboloid approach.

Our idea is to estimate the critical determinant of the body $K_a: \ |f_a| \leq 1$ by inscribing a double paraboloid and using the author's recent result (1.9) in the form that $\Delta(\mathcal{P}_0^{(p)}) = \frac{1}{2p}$ for

$$\mathcal{P}_0^{(p)}: \quad p\,|z| + x^2 + y^2 \leq 1\,, \tag{2.1}$$

$p > 0$ a parameter remaining at our disposition. Since f_a is a symmetric function of its variables u, v, w, it is convenient to use a paraboloid whose axis of rotation is the straight line through the origin with direction vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$. In other words, we submit

5) The numerical values are in fact available with much higher accuracy. We have rounded them to a few decimal places to keep this table in a reasonable format.

$\mathcal{P}_0^{(p)}$ to a rotation which sends $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$. Its matrix is given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} .$$

Viewing this as a change of the coordinate system, i.e.,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix} ,$$

we get

$$x = \frac{1}{\sqrt{2}}(u - v) ,$$

$$y = \frac{1}{\sqrt{6}}(u + v) - \frac{\sqrt{2}}{\sqrt{3}} w ,$$

$$z = \frac{1}{\sqrt{3}}(u + v + w) .$$

Under this unimodular transformation, $\mathcal{P}_0^{(p)}$ becomes

$$\mathcal{P}^{(p)} : \qquad \frac{p}{\sqrt{3}} \, |u + v + w| + \frac{2}{3} \left(u^2 + v^2 + w^2 - (uv + uw + vw)\right) \leq 1 . \tag{2.2}$$

We put for short

$$S = u + v + w , \ Q = u^2 + v^2 + w^2 , \ B = uv + uw + vw ,$$

then (2.2) simply reads $\mathcal{P}^{(p)} : \ \frac{p}{\sqrt{3}}|S| + \frac{2}{3}(Q - B) \leq 1$, and

$$f_a = f_a(u, v, w) = u^3 + v^3 + w^3 + 3auvw = S^3 - 3SB + 3(1 + a)uvw ,$$

as a straightforward computation verifies.

Our task is to determine the (absolute) maximum of $|f_a(u, v, w)|$ on the surface $\frac{p}{\sqrt{3}}|S| + \frac{2}{3}(Q - B) = 1$ which obviously is found among the relative extrema of f_a on this set. By symmetry, we may restrict our search to

$$\frac{p}{\sqrt{3}} S + \frac{2}{3}(Q - B) = 1 , \quad S \geq 0 . \tag{2.3}$$

Using this along with the identity $S^2 = Q + 2B$, we see after a quick computation that f_a simplifies to

$$\frac{3\,p}{2} \sqrt{3}(1 - Q) + \frac{3}{2} \, (1 - p^2) S + 3b \, uvw$$

with $b = a + 1$ for short. Since the case $a = -1$ has been settled by (1.3), we shall assume throughout the sequel that $b \neq 0$. We shall thus optimize $cuvw - Q + \alpha S$, where $c := \frac{2b}{p\sqrt{3}}$, $\alpha := \frac{1-p^2}{p\sqrt{3}}$, under the constraint (2.3), by means of Lagrange's rule.[6] Our Lagrange function reads

$$L = L(u, v, w) = cuvw - Q + \alpha S + t\left(\frac{p}{\sqrt{3}}S + \frac{2}{3}(Q - B) - 1\right),$$

thus we get

$$\frac{\partial L}{\partial u} = cvw - 2u + \alpha + t\left(\frac{p}{\sqrt{3}} + \frac{2}{3}(2u - v - w)\right) = 0\,, \tag{2.4}$$

$$\frac{\partial L}{\partial v} = cuw - 2v + \alpha + t\left(\frac{p}{\sqrt{3}} + \frac{2}{3}(2v - u - w)\right) = 0\,, \tag{2.5}$$

$$\frac{\partial L}{\partial w} = cuv - 2w + \alpha + t\left(\frac{p}{\sqrt{3}} + \frac{2}{3}(2w - u - v)\right) = 0\,. \tag{2.6}$$

We claim that there exists no solution of (2.3) – (2.6) with $u \neq v \neq w \neq u$. Assuming the contrary, we subtract (2.5) from (2.4) and divide by $u - v$ to get

$$-cw + 2(t - 1) = 0\,. \tag{2.7}$$

Similarly, from (2.4) and (2.6), after division by $u - w$,

$$-cv + 2(t - 1) = 0\,. \tag{2.8}$$

Subtracting these last two equations yields the contradiction $v = w$.

Thus there remain just two cases (apart from permutations of the variables).

Case 1. Solutions of (2.3) – (2.6) with $u = v = w$. From (2.3) we immediately obtain

$$u_0 = v_0 = w_0 = \frac{1}{p\sqrt{3}}\,, \quad f_a(u_0, v_0, w_0) = \frac{b}{p^3\sqrt{3}}\,. \tag{2.9}$$

Case 2. Solutions of (2.3) – (2.6) with $u = v \neq w$. The deduction of (2.8) remains valid, thus $t = \frac{1}{2}cu + 1$. Inserting this into (2.4) and solving for w, we get[7]

$$w = \frac{6\alpha + \sqrt{3}\,cpu + 2cu^2 + 2\sqrt{3}\,p - 8u}{4(1 - cu)}\,.$$

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[6] We postpone for the moment the possibility of an extremum in the plane $S = 0$.

[7] It is recommendable to carry out this and the subsequent calculations with the support of a symbolic computation package such as Derive [16] and/or Mathematica [18].

Inserting $c = \frac{2b}{p\sqrt{3}}$ and $\alpha = \frac{1-p^2}{p\sqrt{3}}$ again, this becomes[8)]

$$w = w(b,p;u) = \frac{2bu^2 + \sqrt{3}\,pu(b-4) + 3}{2(\sqrt{3}\,p - 2bu)}\,. \tag{2.10}$$

We use this (along with $u = v$) in (2.3) to obtain after substantial simplifications

$$P(b,p;u) := \; -12\,b^2\,u^4 + 8\,\sqrt{3}\,(3-b)\,b\,p\,u^3 + \left(-36\,p^2 + b^2\,\left(8+p^2\right) + 6\,b\,\left(-2+3\,p^2\right)\right)\,u^2 +$$

$$+\sqrt{3}\,p\,\left(12 - b\,\left(8+p^2\right)\right)\,u + 3\,\left(-1+p^2\right) = 0\,.$$

Defining, for given b, p, the finite set

$$\mathcal{S}(b,p) = \begin{cases} \{u \in \mathbb{R}\,:\; P(b,p;u) = 0,\; 2u + w(b,p;u) > 0\,\} & \text{if } b \neq \frac{3p^2}{p^2+2}, \\ \{\,\frac{\sqrt{3}}{3p},\; \frac{\sqrt{3}\,(1-p^2)}{3p}\;\} & \text{if } b = \frac{3p^2}{p^2+2}, \end{cases} \tag{2.11}$$

we see that for this case the maximum of $|f_a|$ is given by

$$\mu_1(b,p) := \max_{u\in\mathcal{S}(b,p)} \left|2u^3 + w(b,p;u)^3 + 3(b-1)u^2\,w(b,p;u)\right| \qquad (b = a+1)\,. \tag{2.12}$$

It remains to determine the extrema of f_a on the circle which is determined by the intersection of the double paraboloid $\frac{p}{\sqrt{3}}|S| + \frac{2}{3}(Q-B) = 1$ with the plane $S = 0$. By this last identity, f_a simplifies to $-3b(u^2v + uv^2)$. The equation of the paraboloid becomes

$$2(u^2 + uv + v^2) = 1\,. \tag{2.13}$$

Thus we get a Lagrange function

$$L = -3b(u^2v + uv^2) + t\left(2(u^2+uv+v^2) - 1\right)\,,$$

and, therefore,

$$\frac{\partial L}{\partial u} = -3b(2uv + v^2) + t(4u + 2v) = 0\,, \tag{2.14}$$

$$\frac{\partial L}{\partial v} = -3b(u^2 + 2uv) + t(2u + 4v) = 0\,. \tag{2.15}$$

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8) To be quite rigorous, we have to discuss the possibility that in (2.10) both nominator and denominator vanish. This would imply that $u = \frac{\sqrt{3}\,p}{2b}$ and $b = \frac{3p^2}{p^2+2}$, hence $u = v = \frac{\sqrt{3}\,(p^2+2)}{6p}$, $w = \frac{\sqrt{3}\,(4-p^2)}{12p}$. These values do not satisfy (2.3). For $b = \frac{3p^2}{p^2+2}$, the polynomial $P(b,p;u)$ possesses exactly 3 roots, namely $\frac{\sqrt{3}}{3p}$, $\frac{\sqrt{3}\,(1-p^2)}{3p}$, and $\frac{\sqrt{3}\,(p^2+2)}{6p}$ (double). We shall take into account this matter when defining the set $\mathcal{S}(b,p)$ below.

Subtracting these two equations, we obtain

$$(u-v)(3b(u+v)+2t)=0\,,$$

hence either $u=v$ or $t=-\frac{3}{2}\,b(u+v)$. For $u=v$, eq. (2.13) readily gives the two solutions $u=v=\pm\frac{1}{\sqrt{6}}$. Inserting $t=-\frac{3}{2}\,b(u+v)$ into (2.14) yields

$$-3b(u+2v)(2u+v)=0\,,$$

hence (since $b\neq 0$) $u=-2v$ or $v=-2u$. In view of (2.13), this gives the four solutions $(\pm\frac{1}{\sqrt{6}},\mp\frac{2}{\sqrt{6}})$, $(\pm\frac{2}{\sqrt{6}},\mp\frac{1}{\sqrt{6}})$. Obviously $|f_a|=\frac{|b|}{\sqrt{6}}$ for all of these altogether six solutions (u,v). We can summarize the results of our analysis as follows.

Lemma. *Let b and p be any real numbers, $b\neq 0$, $p>0$. Then the maximum of*

$$|f_{b-1}(u,v,w)|=\left|u^3+v^3+w^3+3(b-1)uvw\right|$$

on the double paraboloid

$$\mathcal{P}^{(p)}:\quad \frac{p}{\sqrt{3}}|u+v+w|+\frac{2}{3}(u^2+v^2+w^2-(uv+uw+vw))\leq 1$$

is given by

$$\mu^*(b,p)=\max\left(\frac{|b|}{p^3\sqrt{3}},\ \mu_1(b,p),\ \frac{|b|}{\sqrt{6}}\right),$$

where $\mu_1(b,p)$ is defined by (2.12).

In other words, for any $p>0$, $\mathcal{P}^{(p)}$ is contained in the star body $\mu^*(b,p)^{1/3}K_{b-1}$. Recalling that the critical determinant of $\mathcal{P}^{(p)}$ is $\frac{1}{2p}$, it follows that $\mu^*(b,p)\Delta(K_{b-1})\geq \Delta(\mathcal{P}^{(p)})=\frac{1}{2p}$, hence $M_a=\frac{1}{\Delta(K_a)}\leq 2p\,\mu^*(a+1,p)$.

Theorem. *The minimum M_a of the ternary cubic form $u^3+v^3+w^3+3a\,uvw$ satisfies*

$$M_a\leq 2p\,\max\left(\frac{|a+1|}{p^3\sqrt{3}},\ \mu_1(a+1,p),\ \frac{|a+1|}{\sqrt{6}}\right),\tag{2.16}$$

where $p>0$ is an arbitrary real parameter and $\mu_1(a+1,p)$ is defined by (2.12).

It is clear by construction, that the right hand side of (2.16) can be evaluated, for any given a and p, by a well-defined algorithm involving the zeros of a biquadratic polynomial. This can be safely done by a package like Mathematica [18]; using its built-in `FindMinimum`-command, one can find for each a an optimal value of p which makes the upper bound obtained small.

Comparing our result with the bounds exhibited in section 1, we see that our "paraboloid approach" supersedes all previous estimates in the range $-2.31788\ldots < a < -0.48403\ldots$, except for the value $a = -1$, where (1.3) is much stronger (and in fact best possible). We illustrate this by a table indicating the new bounds for M_a provided by our Theorem, along with the corresponding optimal values for p, and the weaker bounds obtained by the "ellipsoid approach" [13], resp., the Mordell-Golser method [8].

a	$M_a \leq$ [new]	p	$M_a \leq$ [13]	$M_a \leq$ [8]
-2.3	2.08141	0.849235	2.08243	2.33665
-2.2	2.00216	0.831914	2.00872	2.26327
-2.1	1.92381	0.812551	1.93567	2.19004
-2	1.84647	0.790795	1.86334	2.11704
-1.9	1.77026	0.766192	1.79185	2.04435
-1.8	1.69532	0.738166	1.72130	1.97210
-1.7	1.62183	0.705962	1.65186	1.90042
-1.6	1.54999	0.668569	1.58370	1.82951
-1.5	1.48007	0.624567	1.51705	1.77074
-1.4	1.41242	0.571851	1.45223	1.71431
-1.3	1.34751	0.507026	1.38964	1.65884
-1.2	1.28602	0.423765	1.32989	1.60466
-1.1	1.22914	0.306503	1.27403	1.55216
-0.9	1.29904	1	1.41421	1.45416
-0.8	1.29904	1	1.41421	1.40983
-0.7	1.29904	1	1.41421	1.36948
-0.6	1.29904	1	1.41421	1.33379
-0.5	1.29904	1	1.41421	1.30337

Concerning the last five lines of this table, a bit of explanation seems appropriate: For $p = 1$ (which the numerical calculation recommends as the optimal value), it is clear that $u = 0$ is a zero of $P(b, 1; u)$, independently of b. Accordingly, by (2.10), $w = \frac{\sqrt{3}}{2}$, and $2f_a(0, 0, \frac{\sqrt{3}}{2}) = \frac{3}{4}\sqrt{3} \approx 1.29904$ which is equal to $2\mu^*(a+1, 1)$ for $-0.9 \leq a \leq -0.5$.

3. Appendix. Remarks on general ternary cubic forms. The important condition that the form $\mathfrak{f}(u,v,w)$ be regular has frequently been omitted in the literature, as far as the Geometry of Numbers is concerned. (To cite just one bad example we refer to the author's previous paper [13].) Therefore, we discuss the matter in some detail.

In fact, the assumption that

$$(*) \qquad \left(\frac{\partial \mathfrak{f}}{\partial u}, \frac{\partial \mathfrak{f}}{\partial v}, \frac{\partial \mathfrak{f}}{\partial w}\right) = (0,0,0) \qquad \text{only for } (u,v,w) = (0,0,0)$$

is usually expressed as: "The discriminant of $\mathfrak{f}$ is nonzero". To understand what this discriminant is, we suppose that there exists some nontrivial solution (u,v,w) (with $w \neq 0$, say) and rewrite $(*)$ in the shape

$$\mathfrak{f}_1(t_1,t_2,1) = \mathfrak{f}_2(t_1,t_2,1) = \mathfrak{f}_3(t_1,t_2,1) = 0 .$$

(Here we used homogeneity and put $t_1 = \frac{u}{w}$, $t_2 = \frac{v}{w}$, $\mathfrak{f}_1 = \frac{\partial \mathfrak{f}}{\partial u}$, etc.). In principle it is possible to eliminate t_1, t_2 from this 3 polynomial equations and arrive at an equality

$$\text{polynomial in the coefficients of } \mathfrak{f} = 0$$

whose left-hand side essentially is the discriminant. To gain a bit more insight into its explicit nature, one can appeal to a very old article of Hesse [11]. According to his "Lehrsatz 4"[9], one can proceed as follows: Let $\varphi = \det(\mathfrak{f}_{ij}) = \frac{\partial(\mathfrak{f}_1,\mathfrak{f}_2,\mathfrak{f}_3)}{\partial(u,v,w)}$ denote the Hessian of $\mathfrak{f}$, and φ_1, φ_2, φ_3 its partial derivatives of first order. Clearly, $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, φ_1, φ_2, φ_3 are homogeneous quadratic polynomials in u,v,w. Let $\mathcal{M}$ denote the (6×6)-matrix which contains (row by row) the coefficients of $u^2, v^2, w^2, uv, uw, vw$ in these 6 polynomials. Then, as shown in Hesse [11], $(*)$ has nontrivial solutions if and only if $\det \mathcal{M} = 0$. Thus $\det \mathcal{M}$ defines (up to a numerical factor, which is a matter of convention anyway) the *discriminant* of the form $\mathfrak{f}$. It is not difficult to implement the above program in the syntax of Mathematica [18]. For instance, for the special forms $\mathfrak{f}:\ a_1u^3 + a_2v^3 + a_3w^3 + 3auvw$ one obtains

$$|\det \mathcal{M}| = 2^9\, 3^{12}\, a_1a_2a_3(a_1a_2a_3 + a^3)^3 .$$

Thus such an $\mathfrak{f}$ is regular iff $0 \neq a_1a_2a_3 \neq -a^3$ (cf. also the condition $a \neq -1$ in (1.1)).

Concerning the forms with vanishing discriminant, in fact several cases have to be distinguished. These may be found in a classic article of Poincaré [15]. It can be shown that there always exists a real non-singular transformation which leads to one of the following canonical forms (b some nonzero constant throughout):

$$u^3 + v^3 + buvw , \tag{4.1}$$

$$(u^2+v^2)w + bu(u^2 - 3v^2) , \tag{4.2}$$

[9] In its first line, "vom dritten Grade" obviously should read "vom zweiten Grade".

$$w^3 + buv^2\,, \tag{4.3}$$

$$w^3 + buvw\,, \tag{4.4}$$

$$w^3 + b(u^2 + v^2)w\,, \tag{4.5}$$

$$vw^2 + buv^2\,. \tag{4.6}$$

To this list one has to add only those canonical forms which split into three linear factors, i.e. uvw and $(u^2 + v^2)w$ (they have been dealt with by DAVENPORT, [2], [3]), and the *degenerate* forms which contain less than 3 variables. For these latter forms an obvious application of Minkowski's Convex Body Theorem shows that their minimum equals 0.

However, for the forms (4.1) – (4.6) no results concerning their minima (in the sense of the Geometry of Numbers) seem to exist in the literature.

References

[1] DAVENPORT, H.: On the product of three homogeneous linear forms, *J. London Math. Soc.* **13** (1938), 139-145.

[2] DAVENPORT, H.: On the product of three homogeneous linear forms (II), *Proc. London Math. Soc.* (2) **44** (1938), 412-431.

[3] DAVENPORT, H.: On the product of three homogeneous linear forms (III), *Proc. London Math. Soc.* (2) **45** (1939), 98-125.

[4] DAVENPORT, H.: Note on the product of three homogeneous linear forms, *J. London Math. Soc.* **16** (1941), 98-101.

[5] DAVENPORT, H.: On the product of three homogeneous linear forms (IV), *Proc. Cambridge Phil. Soc.* **39** (1943), 1-21.

[6] DAVENPORT, H.: On the minimum of a ternary cubic form, *J. London Math. Soc.* **19** (1944), 13-18.

[7] DAVENPORT, H.: On the minimum of $X^3 + Y^3 + Z^3$, *J. London Math. Soc.* **21** (1946), 82-86.

[8] GOLSER, G.: Schranken für Gitterkonstanten einiger Sternkörper. Ph. D. Dissertation, University of Vienna, 1973.

[9] GOLSER, G.: Über Gitterkonstanten spezieller Sternkörper, *Sitz.-Ber. Österr. Akad. Wiss., Abt. II, math.-naturwiss. Kl.,* **184** (1975), 227-238.

[10] GRUBER, P.M., and LEKKERKERKER, C.G.: Geometry of numbers. North Holland, Amsterdam-New York-Oxford-Tokyo, 1987.

[11] HESSE, O.: Über die Elimination der Variablen aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variablen, *J. reine angew. Math.* **28** (1844), 68-96.

[12] MORDELL, L.J.: On the minimum of a ternary cubic form, *J. London Math. Soc.* **19** (1944), 6-12.

[13] NOWAK, W.G.: On the minimum of a ternary cubic form, *Fibonacci Q.* **24** (1986), 129-132.

[14] NOWAK, W.G.: The critical determinant of the double paraboloid and Diophantine approximation in $\mathbb{R}^3$ and $\mathbb{R}^4$, *Math. Pannonica* **10** (1999), 111-122.

[15] POINCARE, H.: Sur les formes cubiques ternaires et quaternaires, *J. de l'École Polytechnique* **50** (1881), 190-253.

[16] SOFT WAREHOUSE, Inc., *Derive*, Version 3.11, Soft Warehouse, Inc., Honolulu (Hawaii) 1995.

[17] WEBER, H.: Lehrbuch der Algebra, 3rd ed., vol. II, Chelsea Publ. Co., New York (no year).

[18] WOLFRAM Research, Inc.: *Mathematica*, Version 3.0. Wolfram Research, Inc., Champaign, 1998.