

**The average number of solutions
of the Diophantine equation $U^2 + V^2 = W^3$
and related arithmetic functions**

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Abstract. For the number of integer solutions of the title equation, with $W \leq x$ (x a large parameter), an asymptotics of the form $Ax \log x + Bx + O\left(x^{1/2}(\log x)^3(\log \log x)^2\right)$ is established. This is achieved in a general setting which furnishes applications to some other natural arithmetic functions.

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1. Introduction. During the problem session of the 1991 Czechoslovak Number Theory Conference, A. Schinzel proposed the following question: Let $r(n)$ denote the number of ways to write the positive integer n as a sum of two squares, and consider the asymptotic formula

$$\sum_{n \leq x} (r(n))^2 = 4x \log x + Cx + O\left(x^{1/2}\omega(x)\right), \quad (1.1)$$

as $x \rightarrow \infty$. How small can the factor $\omega(x)$ be made, if one uses the sharpest tools of contemporary analytic number theory?

In fact, asymptotics of the shape⁽¹⁾ (1.1) had been established by B.M. Wilson [19], with $\omega(x) = x^\varepsilon$, and W. Recknagel [13], with $\omega(x) = (\log x)^6$.

Subsequently, as a reaction to Schinzel's proposal, the first named author [6] sharpened the estimate to $\omega(x) = (\log x)^{11/3}(\log \log x)^{1/3}$.

The problem addressed by Schinzel is closely related to another arithmetic question to which our present title refers: Given a large parameter x , how many integer triples (u, v, w) exist with $u^2 + v^2 = w^3$, $w \leq x$? This matter was dealt with by K.H. Fischer [2] and also by Recknagel [13].

Since $\frac{1}{4}r(\cdot)$ is multiplicative, it is easy to write up the corresponding generating Dirichlet series for $(r(n))^2$ and $r(n^3)$. (See section 4.2 for details.) Thus, both problems are subsumed in a natural way by the following more general result which is to be the objective of the present note.

⁽¹⁾ To complete the history, one should mention the older, somewhat coarser results of W. Sierpiński [16] and S. Ramanujan [12], with error terms $O\left(x^{3/4} \log x\right)$, resp., $O\left(x^{3/5+\varepsilon}\right)$. Furthermore, Schinzel [15] himself had bounded the remainder from below, showing that it is an $\Omega(x^{3/8})$.

Theorem. Let $a(n)$ be an arithmetic function satisfying $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$, with a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{(\zeta_{\mathbb{K}}(s))^2}{(\zeta(2s))^{m_1} (\zeta_{\mathbb{K}}(2s))^{m_2}} G(s) \quad (\Re(s) > 1),$$

where $\zeta_{\mathbb{K}}$ is the Dedekind zeta-function of some quadratic number field \mathbb{K} , $G(s)$ is holomorphic and bounded in some half-plane $\Re(s) \geq \theta$, $\theta < \frac{1}{2}$, and m_1, m_2 are nonnegative integers. Then, for x large,

$$\begin{aligned} \sum_{n \leq x} a(n) &= \operatorname{Res}_{s=1} \left(F(s) \frac{x^s}{s} \right) + O \left(x^{1/2} (\log x)^3 (\log \log x)^{m_1+m_2} \right) = \\ &= Ax \log x + Bx + O \left(x^{1/2} (\log x)^3 (\log \log x)^{m_1+m_2} \right). \end{aligned}$$

Remark. It is natural to ask why we have chosen just this very degree of generality, resp., specialization, in our suppositions. Note that, for \mathbb{K} a quadratic field, $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi)$, where χ is a certain real Dirichlet character. Thus the above function F is a special case of

$$\frac{L(s, \chi^{(1)}) \dots L(s, \chi^{(J)})}{\zeta_{\mathbb{K}_1}(2s) \dots \zeta_{\mathbb{K}_M}(2s)} G(s),$$

where the $L(s, \chi^{(j)})$ are ordinary Dirichlet L -functions, $\mathbb{K}_1, \dots, \mathbb{K}_M$ are arbitrary algebraic number fields, and $G(s)$ is as before. As long as $J = 4$, our argument would go through without major alterations and lead to the same result, apart from the exponents of the log- and loglog-factors in the error term.

However, for $J \neq 4$, the situation changes drastically: If $J = 2$ or 3 , an important rôle is played by the zero-free region of the denominator.⁽²⁾ Using the sharpest information available of this kind, one obtains a bound $O(x^{1/2} \exp(-c(\log x)^{3/5} (\log \log x)^{-1/5}))$ for the remainder, where the exp-factor is familiar from the prime number theorem. (See, for instance, formulas (14.28) – (14.30) and (1.105) – (1.107) in Ivić [5].)

Opposed to this, for $J > 4$ the size of the nominator in vertical strips becomes important. This ultimately leads to an estimate $O(x^{\alpha_J + \varepsilon})$, where α_J is the best known error-exponent in the J -dimensional Piltz divisor problem which, for $J \geq 5$, is still a matter of small improvements from time to time (cf. Ivić [5], p. 355, or Titchmarsh [18], ch. 12). Thus the case $J = 4$ remains as the most delicate one, at least as far as smaller factors, apart from $x^{1/2}$, in the error term are concerned.

In addition, our analysis has been inspired by recent work⁽³⁾ of K. Ramachandra and A. Sankaranarayanan [11] which deals with

⁽²⁾ Therefore, in these cases, the assumption of the truth of the Riemann Hypothesis (RH) leads to a better error term $O(x^\theta)$, with some $\theta < \frac{1}{2}$. In contrast, our Theorem is as sharp as it would be if RH could be proven.

⁽³⁾ The authors are indebted to Professor Sankaranarayanan for sending them a preprint of this paper.

$$\sum_{n=1}^{\infty} \frac{(d(n))^2}{n^s} = \frac{(\zeta(s))^4}{\zeta(2s)} \quad (\Re(s) > 1).$$

See also Sankaranarayanan's Oberwolfach lecture [14]. However, our argument is technically a bit simpler.

2. Some auxiliary results.

Lemma. *Let \mathbb{K} be an arbitrary number field of degree $[\mathbb{K} : \mathbb{Q}] \geq 1$, $\zeta_{\mathbb{K}}$ its Dedekind zeta-function, and $\varepsilon > 0$ fixed. Then, for each sufficiently large T there exists a measurable set $\mathcal{A}_T = \mathcal{A}_{T, \mathbb{K}} \subset [T, 2T]$ of Lebesgue measure*

$$\lambda(\mathcal{A}_T) \leq T^\varepsilon, \quad (2.1)$$

with the property that, for some $C_{\varepsilon, \mathbb{K}} > 0$ depending only on ε and \mathbb{K} ,

$$\sup_{t \in [T, 2T] \setminus \mathcal{A}_T} |\zeta_{\mathbb{K}}(1 + it)|^{-1} \leq C_{\varepsilon, \mathbb{K}} \log \log T. \quad (2.2)$$

Proof. For the case of the Riemann zeta-function, this is contained in Ramachandra [10], Theorem 1. Our argument follows the lines of Lemma 3.2 in Ramachandra and Sankaranarayanan [11].

According to Heath-Brown [3], there exists some $\delta = \delta(\varepsilon) > 0$ such that, for T large, the cardinality of the set

$$\mathcal{R}_{\varepsilon, T} = \{ \rho \in \mathbb{C} : \zeta_{\mathbb{K}}(\rho) = 0, \Re(\rho) \geq 1 - 3\delta, T \leq \Im(\rho) \leq 2T \} \quad (2.3)$$

satisfies

$$\#\mathcal{R}_{\varepsilon, T} \ll T^{\varepsilon/2}. \quad (2.4)$$

For any fixed $c > 0$, we define

$$\mathcal{A}(c, T) := [T, T + cT^{\varepsilon/4}] \cup [2T - cT^{\varepsilon/4}, 2T] \cup \bigcup_{\rho \in \mathcal{R}_{\varepsilon, T}} [\Im(\rho) - cT^{\varepsilon/4}, \Im(\rho) + cT^{\varepsilon/4}],$$

and

$$\mathcal{M}(c, T) := \{s \in \mathbb{C} : \Re(s) \geq 1 - 3\delta, \Im(s) \in [T, 2T] \setminus \mathcal{A}(c, T)\} \cup \{s \in \mathbb{C} : \Re(s) \geq 1\}. \quad (2.5)$$

On every $\mathcal{M}(c, T)$, by construction $\zeta_{\mathbb{K}}(s) \neq 0$, thus $\log \zeta_{\mathbb{K}}(s)$ can be defined properly by analytic continuation⁽⁴⁾. For any $\eta = 2 + it$ with $t \in [T, 2T] \setminus \mathcal{A}(1, T)$, we consider the circular discs

$$\mathcal{C}_1 := \{s \in \mathbb{C} : |s - \eta| \leq 1 + \delta\}, \quad \mathcal{C}_2 := \{s \in \mathbb{C} : |s - \eta| \leq 1 + 2\delta\}.$$

⁽⁴⁾ To be explicit, on $\mathcal{M}(c, T)$, we can define $\log \zeta_{\mathbb{K}}(s) := \log \zeta_{\mathbb{K}}(2) + \int_{\mathcal{C}} (\zeta'_{\mathbb{K}}(z)/\zeta_{\mathbb{K}}(z)) dz$ where \mathcal{C} consists of the two straight line segments from 2 to $2 + \Im(s)i$ and from $2 + \Im(s)i$ to s .

Evidently, $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{M}(\frac{1}{2}, T)$. Thus the Borel-Carathéodory inequality (cf. Titchmarsh [17], ch. 5.5) yields

$$\max_{s \in \mathcal{C}_1} |\log \zeta_{\mathbb{K}}(s)| \leq \frac{2(1+\delta)}{\delta} \max_{s \in \mathcal{C}_2} \log |\zeta_{\mathbb{K}}(s)| + \frac{2+3\delta}{\delta} |\log \zeta_{\mathbb{K}}(2+it)|.$$

Since, for a certain $C > 0$, $|\zeta_{\mathbb{K}}(\sigma+it)| \leq |t|^C$, provided that $\sigma \geq \frac{1}{2}$, $|t|$ sufficiently large, it follows that $\log |\zeta_{\mathbb{K}}(s)| \leq C \log(2T)$ on \mathcal{C}_2 . Hence, in particular,

$$|\log \zeta_{\mathbb{K}}(\sigma+it)| \ll \log T \quad \text{for } \sigma \geq 1-\delta,$$

the constant involved not depending⁽⁵⁾ on t . Therefore,

$$\log \zeta_{\mathbb{K}}(s) \ll \log T \tag{2.6}$$

uniformly for all s with $\Re(s) \geq 1-\delta$, $\Im(s) \in [T, 2T] \setminus \mathcal{A}(1, T)$.

We now use the series representation (valid for $\Re(s) > 1$)

$$\log \zeta_{\mathbb{K}}(s) = \sum_{\mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{N}(\mathcal{P})^{-ms} = \sum_{\mathcal{P}} \mathcal{N}(\mathcal{P})^{-s} + H(s), \tag{2.7}$$

where $H(s)$ is regular and bounded in any half-plane $\Re(s) \geq \theta > \frac{1}{2}$. (\mathcal{P} are the prime ideals in the ring $\mathcal{O}_{\mathbb{K}}$ of algebraic integers in the field \mathbb{K} and $\mathcal{N}(\cdot)$ denotes the norm of ideals in $\mathcal{O}_{\mathbb{K}}$.)

We put $X = (\log T)^{1/\delta}$. Then, for any $t \in \mathbb{R}$,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw = \\ &= \sum_{\mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{N}(\mathcal{P})^{-m(1+it)} \left(\frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} (X^{-1} \mathcal{N}(\mathcal{P})^m)^{-w} \Gamma(w) dw \right) = \\ &= \sum_{\mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{N}(\mathcal{P})^{-m(1+it)} \exp(-\mathcal{N}(\mathcal{P})^m/X) = \sum_{\mathcal{P}} \mathcal{N}(\mathcal{P})^{-1-it} \exp(-\mathcal{N}(\mathcal{P})/X) + O(1). \end{aligned} \tag{2.8}$$

To estimate the remaining sum, we observe that

$$\begin{aligned} \sum_{\mathcal{N}(\mathcal{P}) > X} \mathcal{N}(\mathcal{P})^{-1-it} \exp(-\mathcal{N}(\mathcal{P})/X) &\ll \sum_{\mathcal{N}(\mathcal{I}) > X} \mathcal{N}(\mathcal{I})^{-1} \exp(-\mathcal{N}(\mathcal{I})/X) = \\ &= \int_X^{\infty} \frac{1}{u} e^{-u/X} d \left(\sum_{\mathcal{N}(\mathcal{I}) \leq u} 1 \right) \ll 1, \end{aligned}$$

⁽⁵⁾ Note that, by (2.7), $|\log \zeta_{\mathbb{K}}(2+it)| \leq \log \zeta_{\mathbb{K}}(2)$.

using only that $\sum_{\mathcal{N}(\mathcal{I}) \leq u} 1 \ll u$. (\mathcal{I} denotes arbitrary integral ideals in $\mathcal{O}_{\mathbb{K}}$.) Further,

$$\left| \sum_{\mathcal{N}(\mathcal{P}) \leq X} \mathcal{N}(\mathcal{P})^{-1-it} \exp(-\mathcal{N}(\mathcal{P})/X) \right| \leq \sum_{\mathcal{N}(\mathcal{P}) \leq X} \mathcal{N}(\mathcal{P})^{-1} = \log \log X + O(1),$$

where the last conclusion is immediate from the prime ideal theorem in the form (cf. Narkiewicz [8], pp. 369-372)

$$\sum_{\mathcal{N}(\mathcal{P}) \leq u} 1 = \frac{u}{\log u} + O\left(\frac{u}{(\log u)^2}\right).$$

In view of (2.8) and the choice $X = (\log T)^{1/\delta}$, this implies that

$$\left| \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw \right| \leq \log \log \log T + O(1). \quad (2.9)$$

Our next step is to put $W = (\log \log T)^2$ and to recall Stirling's formula in the weak form (valid uniformly in any strip $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \geq 1$)

$$\Gamma(\sigma + it) \ll |t|^{\sigma-1/2} \exp(-\frac{\pi}{2}|t|). \quad (2.10)$$

From this it readily follows that

$$\int_{\delta \pm iW}^{\delta \pm i\infty} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw \ll X^\delta \int_W^\infty \exp(-\frac{\pi}{2}u) du \ll X^\delta \exp(-\frac{\pi}{2}W) \ll 1,$$

hence

$$\left| \frac{1}{2\pi i} \int_{\delta-iW}^{\delta+iW} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw \right| \leq \log \log \log T + O(1). \quad (2.11)$$

From now on we impose the condition that $t \in [T, 2T] \setminus \mathcal{A}(2, T)$. We evaluate the integral in (2.11) by the residue theorem, applied to the rectangle \mathfrak{R} with vertices $\pm\delta \pm iW$. (Note that for $t \in [T, 2T] \setminus \mathcal{A}(2, T)$ and $w \in \mathfrak{R}$, necessarily $1+it+w \in \mathcal{M}(1, T)$, and for $\log(\zeta_{\mathbb{K}}(1+it+w))$ the bound (2.6) applies.) We obtain

$$\int_{-\delta-iW}^{-\delta+iW} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw \ll X^{-\delta} \log T \int_{-W}^W |\Gamma(-\delta+iu)| du \ll 1,$$

and similarly, for the horizontal segments,

$$\int_{-\delta \pm iW}^{\delta \pm iW} \log \zeta_{\mathbb{K}}(1+it+w) \Gamma(w) X^w dw \ll X^\delta \log T \exp(-\frac{\pi}{2}W) \ll 1.$$

Since the only pole inside the rectangle, at $w = 0$, gives a residue of $\log \zeta_{\mathbb{K}}(1 + it)$, we altogether derive from (2.11) that

$$|\log \zeta_{\mathbb{K}}(1 + it)| \leq \log \log \log T + C_1,$$

for all $t \in [T, 2T] \setminus \mathcal{A}(2, T)$ ($C_1 > 0$ an appropriate constant depending on \mathbb{K} and ε). Hence, for the same t ,

$$|\zeta_{\mathbb{K}}(1 + it)|^{-1} = \exp(-\log |\zeta_{\mathbb{K}}(1 + it)|) \leq \exp(|\log \zeta_{\mathbb{K}}(1 + it)|) \leq e^{C_1} \log \log T.$$

Taking $\mathcal{A}_{T, \mathbb{K}} = \mathcal{A}(2, T)$, this just proves clause (2.2) of the Lemma, while (2.1) is immediate by (2.4) and (2.5). \square

We conclude this section by stating some more bounds⁽⁶⁾ for the zeta-functions involved, which will be needed in the proof of the Theorem. First of all, again for every number field \mathbb{K} with $[\mathbb{K} : \mathbb{Q}] \geq 1$,

$$|\zeta_{\mathbb{K}}(\sigma + it)|^{-1} \ll (\log(2 + |t|))^C \tag{2.12}$$

uniformly (at least) in $\sigma \geq 1$, the constant $C > 0$ possibly depending on \mathbb{K} . (This is most conveniently deduced after the classic example of Apostol [1], Th. 13.7, using the necessary facts about the Dedekind zeta-function from Narkiewicz [8], ch. 7.2.)

Further, we recall that

$$\zeta\left(\frac{1}{2} + it\right) \ll (1 + |t|)^{1/6+\varepsilon}, \quad L(-\varepsilon' + it, \chi) \ll_{\chi} (1 + |t|)^{1/2+\varepsilon}$$

for any Dirichlet L -series. (For the first estimate, see Titchmarsh [18], Th. 5.12. The second one follows readily from the functional equation in Apostol [1], Th. 12.11, along with Stirling's formula.) Applying the Phragmén-Lindelöf theorem in the form given by Titchmarsh [17], ch. 5.65, we infer that

$$\zeta(\sigma + it) \ll 1 + |t|^{(1-\sigma)/3+\varepsilon}, \quad L(\sigma + it, \chi) \ll_{\chi} 1 + |t|^{(1-\sigma)/2+\varepsilon},$$

uniformly (at least) in $\sigma \geq \frac{1}{2}$, $|t| \geq 1$. Now, if \mathbb{K} is a *quadratic* field of discriminant D , its zeta-function can be factorized as $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi_D)$. (Cf. Zagier [21], p. 100, in particular for the definition of the real character χ_D involved.) Therefore, in this case,

$$\zeta_{\mathbb{K}}(\sigma + it) \ll 1 + |t|^{\frac{5}{6}(1-\sigma)+\varepsilon}, \tag{2.13}$$

uniformly in $\sigma \geq \frac{1}{2}$, $|t| \geq 1$. Last but not least, it has been proved by W. Müller [7] (even in the stronger form of an asymptotics) that

$$\int_0^T |\zeta_{\mathbb{K}}\left(\frac{1}{2} + it\right)|^2 dt \ll T(\log T)^2, \tag{2.14}$$

⁽⁶⁾ In fact, the estimates (2.12) and (2.13) are far away from being the sharpest ones of their kind, but they suffice for our purpose and are conveniently available in textbooks.

again for every quadratic field \mathbb{K} .

3. Proof of the Theorem. W.l.o.g., let x be half an odd integer. Then, by a simple version of Perron's truncated formula (see, e.g., Prachar [9], p. 376),

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-ix^{3/5}}^{1+\varepsilon+ix^{3/5}} F(s) x^s \frac{ds}{s} + O\left(x^{1/2}\right), \quad (3.1)$$

for any fixed $\varepsilon > 0$ sufficiently small. We shift the path of integration to the vertical line $\Re(s) = \frac{1}{2}$. The horizontal segments contribute

$$\frac{1}{2\pi i} \int_{\frac{1}{2} \pm ix^{3/5}}^{1+\varepsilon \pm ix^{3/5}} F(s) x^s \frac{ds}{s} \ll \int_{\frac{1}{2}}^{1+\varepsilon} \frac{|\zeta_{\mathbb{K}}(\sigma \pm ix^{3/5})|^2}{|\zeta(2\sigma \pm 2ix^{3/5})|^{m_1} |\zeta_{\mathbb{K}}(2\sigma \pm 2ix^{3/5})|^{m_2}} x^{\sigma-3/5} d\sigma$$

By (2.12) and (2.13), this is

$$\ll x^{-3/5+\varepsilon} \max_{\frac{1}{2} \leq \sigma \leq 1+\varepsilon} \left(x^{\sigma} (x^{3/5})^{5(1-\sigma)/3} \right) \ll x^{2/5+\varepsilon}.$$

Hence, by the residue theorem,

$$\sum_{n \leq x} a(n) = \operatorname{Res}_{s=1} \left(F(s) \frac{x^s}{s} \right) + \frac{1}{2\pi i} \int_{\frac{1}{2}-ix^{3/5}}^{\frac{1}{2}+ix^{3/5}} F(s) x^s \frac{ds}{s} + O\left(x^{1/2}\right). \quad (3.2)$$

To estimate the remaining integral, we use the parametrization $s = \frac{1}{2}(1+it)$, $|t| \leq 2x^{3/5}$, along with a dyadic decomposition, to obtain

$$\int_{\frac{1}{2}-ix^{3/5}}^{\frac{1}{2}+ix^{3/5}} F(s) x^s \frac{ds}{s} \ll x^{1/2} \left(1 + \sum_{\substack{T=2^{-j}x^{3/5}, \\ j=0,1,2,\dots}} \frac{1}{T} \int_T^{2T} \frac{|\zeta_{\mathbb{K}}(\frac{1}{2}(1+it))|^2}{|\zeta(1+it)|^{m_1} |\zeta_{\mathbb{K}}(1+it)|^{m_2}} dt \right), \quad (3.3)$$

where the sum in fact contains only $O(\log x)$ terms. With $\mathcal{A}_{\mathbb{K},T}$ as in the Lemma, we put $\mathcal{B}_{\mathbb{K},T} = [T, 2T] \setminus \mathcal{A}_{\mathbb{K},T}$. By (2.1), (2.12), and (2.13),

$$\int_{\mathcal{A}_{\mathbb{K},T}} \frac{|\zeta_{\mathbb{K}}(\frac{1}{2}(1+it))|^2}{|\zeta(1+it)|^{m_1} |\zeta_{\mathbb{K}}(1+it)|^{m_2}} dt \ll T^{5/6+4\varepsilon}.$$

Moreover, by (2.2) and (2.14),

$$\int_{\mathcal{B}_{\mathbb{K},T}} \frac{|\zeta_{\mathbb{K}}(\frac{1}{2}(1+it))|^2}{|\zeta(1+it)|^{m_1} |\zeta_{\mathbb{K}}(1+it)|^{m_2}} dt \ll T(\log T)^2 (\log \log T)^{m_1+m_2}.$$

Inserting the last two bounds into (3.3) and summing over T , we obtain

$$\int_{\frac{1}{2}-ix^{3/5}}^{\frac{1}{2}+ix^{3/5}} F(s) x^s \frac{ds}{s} \ll x^{1/2} (\log x)^3 (\log \log x)^{m_1+m_2}.$$

Together with (3.2), this completes the proof of our Theorem.

4. Applications. It remains to verify that the result established covers the two special problems addressed in the title and in the introduction, even in a more general context.

4.1. The second moment of quadratic Dedekind-zeta coefficients. For an arbitrary quadratic number field \mathbb{K} with discriminant D , let $\mathcal{O}_{\mathbb{K}}$ denote the ring of algebraic integers in \mathbb{K} , and $r_{\mathbb{K}}(n)$ the number of integral ideals \mathcal{I} in $\mathcal{O}_{\mathbb{K}}$ of Norm $\mathcal{N}(\mathcal{I}) = n$. Then, as we shall show below, for $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{(r_{\mathbb{K}}(n))^2}{n^s} = \frac{(\zeta_{\mathbb{K}}(s))^2}{\zeta(2s)} \prod_{p|D} (1+p^{-s})^{-1}. \quad (4.1)$$

Therefore, applying the Theorem with $(m_1, m_2) = (1, 0)$, we obtain what follows⁽⁷⁾.

Corollary 1. *For any quadratic field \mathbb{K} of discriminant D , and x large,*

$$\sum_{n \leq x} (r_{\mathbb{K}}(n))^2 = A_1 x \log x + B_1 x + O\left(x^{1/2} (\log x)^3 \log \log x\right),$$

with

$$A_1 = \frac{6}{\pi^2} L(1, \chi_D)^2 \prod_{p|D} \frac{p}{p+1},$$

$$B_1 = \frac{6}{\pi^2} L(1, \chi_D)^2 \prod_{p|D} \frac{p}{p+1} \left(-1 + 2\gamma + \sum_{p|D} \frac{\log p}{p+1} + \frac{2L'(1, \chi_D)}{L(1, \chi_D)} - \frac{12}{\pi^2} \zeta'(2) \right),$$

where γ is the Euler-Mascheroni constant.

Since $r(n) = 4r_{\mathbb{Q}(i)}(n)$, this implies in particular that

$$\sum_{n \leq x} (r(n))^2 = 4x \log x + 16B_1 x + O\left(x^{1/2} (\log x)^3 \log \log x\right) \quad (4.2)$$

with

$$16B_1 = -4 + 8\gamma + \frac{4}{3} \log 2 + \frac{32}{\pi} L'(1) - \frac{48}{\pi^2} \zeta'(2) \approx 8.0665,$$

where $L(\cdot)$ is the L -function corresponding to the non-principal Dirichlet character mod 4.

⁽⁷⁾ For the evaluation of the residue at $s = 1$ and the subsequent numerical computations, we have employed *Mathematica* [20].

It remains to verify (4.1). To this end, we recall the decomposition laws in $\mathcal{O}_{\mathbb{K}}$ (cf. Zagier [21]). On the set \mathbb{P} of all rational primes p , there exists a partition $\mathbb{P} = \mathbb{P}_0 \cup \mathbb{P}_1 \cup \mathbb{P}_2$ such that⁽⁸⁾

$$(i) \quad p \in \mathbb{P}_0 \iff p|D \iff (p) = \mathcal{P}^2, \mathcal{N}(\mathcal{P}) = p,$$

$$(ii) \quad p \in \mathbb{P}_1 \iff (p) = \mathcal{P}_1\mathcal{P}_2, \mathcal{P}_1 \neq \mathcal{P}_2, \mathcal{N}(\mathcal{P}_1) = \mathcal{N}(\mathcal{P}_2) = p,$$

$$(iii) \quad p \in \mathbb{P}_2 \iff (p) \text{ prime in } \mathcal{O}_{\mathbb{K}}, \mathcal{N}(p) = p^2.$$

For $\Re(s) > 1$, this implies the Euler product representation

$$\begin{aligned} \zeta_{\mathbb{K}}(s) &= \prod_{p \in \mathbb{P}} \left(1 + \sum_{j=1}^{\infty} r_{\mathbb{K}}(p^j) p^{-js} \right) = \prod_{\mathcal{P}} (1 - \mathcal{N}(\mathcal{P})^{-s})^{-1} = \\ &= \prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} (1 - p^{-s})^{-2} \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1}. \end{aligned} \quad (4.3)$$

By comparison, we immediately see that $r_{\mathbb{K}}(p^j) = 1$ throughout for $p \in \mathbb{P}_0$, $r_{\mathbb{K}}(p^j) = j + 1$ for $p \in \mathbb{P}_1$, and $r_{\mathbb{K}}(p^j) = \begin{cases} 1 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$ for $p \in \mathbb{P}_2$.

Hence,

$$\sum_{n=1}^{\infty} \frac{(r_{\mathbb{K}}(n))^2}{n^s} = \prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} \left(1 + \sum_{j=1}^{\infty} (j+1)^2 p^{-js} \right) \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1}.$$

Using that $\sum_{j \geq 0} (j+1)^2 z^j = (1+z)(1-z)^{-3}$ (for $|z| < 1$), we see that this equals

$$\begin{aligned} &\prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} (1 - p^{-s})^{-3} (1 + p^{-s}) \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1} = \\ &= \prod_{p \in \mathbb{P}} (1 - p^{-2s}) \left(\prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} (1 - p^{-s})^{-2} \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1} \right)^2 \prod_{p \in \mathbb{P}_0} (1 + p^{-s})^{-1} = \\ &= \frac{(\zeta_{\mathbb{K}}(s))^2}{\zeta(2s)} \prod_{p|D} (1 + p^{-s})^{-1}, \end{aligned}$$

which proves (4.1).

⁽⁸⁾ Note that, on the basis of this partition, the Dirichlet character χ_D can be defined as follows: $\chi(p) = 0$ if $p \in \mathbb{P}_0$, $\chi(p) = 1$ if $p \in \mathbb{P}_1$, and $\chi(p) = -1$ if $p \in \mathbb{P}_2$.

4.2. Diophantine equations as addressed in the title. Again for \mathbb{K} a quadratic field, we shall show that

$$F(s) := \sum_{n=1}^{\infty} \frac{r_{\mathbb{K}}(n^3)}{n^s} = \frac{(\zeta_{\mathbb{K}}(s))^2}{\zeta(2s)\zeta_{\mathbb{K}}(2s)} G(s) \quad (\Re(s) > 1), \quad (4.4)$$

where $G(s)$ is holomorphic and bounded in every half-plane $\Re(s) \geq \theta > \frac{1}{3}$. Taking this for granted, we apply our Theorem with $(m_1, m_2) = (1, 1)$ and derive the following consequence.

Corollary 2. *For any quadratic field \mathbb{K} of discriminant D , and x large,*

$$\sum_{n \leq x} r_{\mathbb{K}}(n^3) = A_2 x \log x + B_2 x + O\left(x^{1/2}(\log x)^3(\log \log x)^2\right),$$

with

$$A_2 = \frac{36 L(1, \chi_D)^2 G(1)}{\pi^4 L(2, \chi_D)},$$

$$B_2 = \frac{36 L(1, \chi_D)^2 G(1)}{\pi^4 L(2, \chi_D)} \left(\frac{2L'(1, \chi_D)}{L(1, \chi_D)} - 1 + 2\gamma + \frac{G'(1)}{G(1)} - \frac{2L'(2, \chi_D)}{L(2, \chi_D)} - \frac{24\zeta'(2)}{\pi^2} \right),$$

where $G(s)$ is given in form of an Euler product in (4.6) below⁽⁹⁾.

To apply this result to Diophantine equations, let $Q = Q(u, v) = au^2 + buv + cv^2$ be an integral, primitive, positive definite binary quadratic form of class number 1.⁽¹⁰⁾ Then it is well-known (cf. Zagier [21], §§ 8 and 11) that

$$r_Q(n) := \#\{(u, v) \in \mathbb{Z}^2 : Q(u, v) = n\} = \omega_D r_{\mathbb{K}}(n), \quad (4.5)$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, and⁽¹¹⁾ $\omega_D = \begin{cases} 6 & \text{for } D = -3, \\ 4 & \text{for } D = -4, \\ 2 & \text{else.} \end{cases}$

We thus can infer the following conclusion.

Corollary 3. *For every integral, primitive, positive definite binary quadratic form Q of class number 1, with discriminant D , and for large real x ,*

$$\begin{aligned} & \#\{(u, v, w) \in \mathbb{Z}^3 : Q(u, v) = w^3, w \leq x\} = \\ & = 1 + \sum_{1 \leq n \leq x} r_Q(n^3) = 1 + \omega_D \sum_{1 \leq n \leq x} r_{\mathbb{Q}(\sqrt{D})}(n^3) = \\ & = \omega_D (A_2 x \log x + B_2 x) + O\left(x^{1/2}(\log x)^3(\log \log x)^2\right), \end{aligned}$$

where A_2, B_2 are as in Corollary 2.

⁽⁹⁾ Of course, $G'(1)/G(1)$ is most easily evaluated as $\left. \frac{d}{ds} \log G(s) \right|_{s=1}$, which transforms the products into sums of derivatives of logarithms.

⁽¹⁰⁾ Equivalently, the discriminant $D = b^2 - 4ac$ is one of the "numeri idonei" $\{-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163\}$. See, e.g., Hlawka/Schoißengeier [4], p. 92.

⁽¹¹⁾ ω_D is the number of automorphisms of the form Q or, in other terms, the number of units in $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$.

In particular, for the Diophantine equation of the title, we obtain

$$\begin{aligned} \#\{(u, v, w) \in \mathbb{Z}^3 : u^2 + v^2 = w^3, w \leq x\} &= 1 + \sum_{1 \leq n \leq x} r(n^3) = \\ &= 4A_2x \log x + 4B_2x + O\left(x^{1/2}(\log x)^3(\log \log x)^2\right), \end{aligned}$$

with

$$4A_2 = \frac{9G(1)}{\pi^2 \mathbf{C}} \approx 0.9091,$$

$$4B_2 = \frac{9G(1)}{\pi^2 \mathbf{C}} \left(\frac{8L'(1, \chi_{-4})}{\pi} - 1 + 2\gamma + \frac{G'(1)}{G(1)} - \frac{2L'(2, \chi_{-4})}{\mathbf{C}} - \frac{24\zeta'(2)}{\pi^2} \right) \approx 2.1715,$$

where $\mathbf{C} = L(2, \chi_{-4}) = \sum_{j \geq 0} (-1)^j (2j+1)^{-2}$ is Catalan's constant, and, by (4.6) below,

$$G(1) = \frac{8}{9} \prod_{p \equiv 1 \pmod{4}} \frac{p^3(p+2)}{(p-1)(p+1)^3} \prod_{p \equiv 3 \pmod{4}} \frac{p^4}{p^4-1} \approx 0.91317,$$

$$\frac{G'(1)}{G(1)} = -\frac{1}{3} \log 2 - 6 \sum_{p \equiv 1 \pmod{4}} \frac{\log p}{(p+1)(p-1)(p+2)} - 4 \sum_{p \equiv 3 \pmod{4}} \frac{\log p}{p^4-1} \approx -0.35876.$$

It remains to verify (4.4). By what we noted earlier (right after (4.3)), for $\Re(s) > 1$,

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{r_{\mathbb{K}}(n^3)}{n^s} = \prod_{p \in \mathbb{P}} \left(1 + \sum_{j=1}^{\infty} r_{\mathbb{K}}(p^{3j}) p^{-js} \right) = \\ &= \prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} \left(1 + \sum_{j=1}^{\infty} (3j+1) p^{-js} \right) \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1}. \end{aligned}$$

Using that $\sum_{j \geq 0} (3j+1)z^j = (1+2z)(1-z)^{-2}$ (for $|z| < 1$), we may write this as

$$F(s) = \prod_{p \in \mathbb{P}_0} (1 - p^{-s})^{-1} \prod_{p \in \mathbb{P}_1} \frac{1 + 2p^{-s}}{(1 - p^{-s})^2} \prod_{p \in \mathbb{P}_2} (1 - p^{-2s})^{-1}.$$

Recalling the Euler product representation of $\zeta_{\mathbb{K}}(s)$ and $\zeta_{\mathbb{K}}(2s)$, as given in (4.3), along with that of $\zeta(2s)$, we obtain after a brief calculation

$$G(s) = F(s) \frac{\zeta(2s)\zeta_{\mathbb{K}}(2s)}{(\zeta_{\mathbb{K}}(s))^2} = \prod_{p \in \mathbb{P}_0} \frac{1 - p^{-s}}{(1 - p^{-2s})^2} \prod_{p \in \mathbb{P}_1} h(p^{-s}) \prod_{p \in \mathbb{P}_2} (1 - p^{-4s})^{-1}, \quad (4.6)$$

where $h(z) := (1+2z)(1-z)^2(1-z^2)^{-3} = 1 + O(|z|^3)$ around $z = 0$. Thus $G(s)$ possesses a Dirichlet series absolutely convergent for $\Re(s) > \frac{1}{3}$. This proves (4.4).

References

- [1] T.M. APOSTOL, Introduction to analytic number theory. New York, Springer, 1976.
- [2] K.H. FISCHER, Über die Anzahl der Gitterpunkte auf Kreisen mit quadratfreien Radienquadraten. Arch. Math. **33** (1979), 150–154.
- [3] D.R. HEATH-BROWN, On the density of the zeros of the Dedekind zeta-function. Acta Arithm. **33** (1977), 169–181.
- [4] E. HLAWKA & J. SCHOI ENGEIER, Zahlentheorie. 2. Aufl., Wien, Manz, 1990.
- [5] A. IVIC, The Riemann zeta-function. New York, Wiley, 1985.
- [6] M. KUHLEITNER, On a question of A. Schinzel concerning the sum $\sum_{n \leq x} (r(n))^2$. Proc. Österr.-Ung.-Slowak. Koll. Zahlentheorie, Graz-Mariatrost 1992. Grazer Math. Ber. **318** (1993), 63–67.
- [7] W. MULLER, The mean square of the Dedekind zeta-function in quadratic number fields. Math. Proc. Cambridge Phil. Soc. **106** (1989), 403–417.
- [8] W. NARKIEWICZ, Elementary and analytic theory of algebraic numbers. 2nd ed., Warszawa, Springer & PWN (Polish Scientific Publishers), 1990.
- [9] K. PRACHAR, Primzahlverteilung. Berlin, Springer, 1957.
- [10] K. RAMACHANDRA, A large value theorem for $\zeta(s)$. Hardy-Ramanujan J. **18** (1995), 1–9.
- [11] K. RAMACHANDRA & A. SANKARANARAYANAN, On an asymptotic formula of Srinivasa Ramanujan. Acta Arithm., in press.
- [12] S. RAMANUJAN, Some formulae in the analytic theory of numbers. Mess. Math. **45** (1916), 81–84.
- [13] W. RECKNAGEL, Varianten des Gaußschen Kreisproblems. Abh. Math. Sem. Univ. Hamburg **59** (1989), 183–189.
- [14] A. SANKARANARAYANAN, On an asymptotic formula of Srinivas Ramanujan. Conf. "Theory of the Riemann zeta and allied functions", held at Oberwolfach, Sept. 2001, Math. FI Oberwolfach, Report No. 43/2001, p. 15.
- [15] A. SCHINZEL, On an analytic problem considered by Sierpiński and Ramanujan. Proc. Int. Conf. in honour of V. Kubilius, Palanga (Lithuania) 1991, VSP BV Int. Corp., 1992, pp. 165–171.
- [16] W. SIERPINSKI, Sur la sommation de la série $\sum_{a < n \leq b} \tau(n)f(n)$, où $\tau(n)$ signifie le nombre de décompositions du nombre n en une somme de deux carrés de nombres entiers. (In Polish, French summary.) Prace Mat. Fiz. **18** (1908), 1–59.
- [17] E.C. TITCHMARSH, The theory of functions. 2nd ed., Oxford, Univ. Press, 1939.
- [18] E.C. TITCHMARSH, The theory of the Riemann zeta-function. 2nd ed., revised by D.R. Heath-Brown. Oxford, Univ. Press, 1986.
- [19] B.M. WILSON, Proofs of some formulae enunciated by Ramanujan. Proc. London Math. Soc. **21** (1922), 235–255.
- [20] WOLFRAM Research, Inc., Mathematica 4.1. Champaign 2001.
- [21] D.B. ZAGIER, Zetafunktionen und quadratische Körper. Berlin, Springer, 1981.

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