On fractional part sums: A mean-square asymptotics over short intervals

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To Professor Edmund Hlawka on his 85th birthday.

With the author's compliments.

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Abstract. This article is concerned with sums $\mathcal{S}(t)=\sum_n \psi(tf(n/t))$ where ψ denotes, essentially, the fractional part minus $\frac{1}{2}$, f is a C^4-function with $f''\neq 0$ throughout, summation being extended over an interval of order t. We establish an asymptotic formula for $\int_{T-\Lambda}^{T+\Lambda}(\mathcal{S}(t))^2\,\mathrm{d}t$ for any $\Lambda=\Lambda(T)$ growing faster than $\log T$.

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1. Introduction and statement of results. Let a function $f:[a,b]\to\mathbb{R}$ be 4 times continuously differentiable and suppose that $f''\neq 0$ throughout. Further, let ψ denote a row-of-teeth function satisfying

$$\begin{aligned}\psi(w)&=w-[w]-\tfrac{1}{2}\qquad \text{for } w\notin\mathbb{Z}\,,\\ -\tfrac{1}{2}\le\psi(w)&\le\tfrac{1}{2}\qquad \text{for } w\in\mathbb{Z}\,.\end{aligned}\tag{1.1}$$

Finally, let t denote a large real parameter.

Continuing earlier work [16], [13], this article will be devoted to the investigation of sums

$$\mathcal{S}(t)=\sum_{at<n\le bt,\ n\in\mathbb{Z}}\psi\left(tf\left(\frac{n}{t}\right)\right).$$

As was pointed out in [16], these sums are connected with the problem of counting the lattice points in the Euclidean (ξ,η)-plane between the curve

$$\frac{\eta}{t}=f\left(\frac{\xi}{t}\right)$$

and the ξ-axis. (For very well-readable expositions of this theory, the reader is referred to the monographs of Krätzel [14], [15].) In classic times, upper bounds for such fractional part sums have been obtained by Vinogradov [19] and van der Corput [3], [4], who ultimately proved that

$$\mathcal{S}(t)\ll t^{2/3-\delta_0}$$

with some (very small) $\delta_0>0$. Under the general conditions stated, there was no improvement until the last decade of the 20th century, when Huxley developed his

"Discrete Hardy Littlewood Method", on the basis of earlier ideas due to Bombieri and Iwaniec [1] and Iwaniec and Mozzochi [12]. Huxley's sharpest version published to date(1) [9] contains the bound(2)

$$\mathcal{S}(t) \ll t^{46/73} (\log t)^{315/146} . \tag{1.2}$$

For a full account, the reader is referred to Huxley's textbook [11]. A more general sum which contains a second parameter $U \geq t$, namely

$$\mathcal{S}(t, U) = \sum_{at<n\leq bt,\ n\in\mathbb{Z}} \psi\left(U f\left(\frac{n}{t}\right)\right)$$

admits important applications to rounding error estimations in connection with classic numerical integration methods. This, too, has been investigated thoroughly by Huxley [8]. The estimates he obtained depend in a complicated way on the relative size of U and t and are given in a table exceeding one printed page.

To state a lower bound for $\mathcal{S}(t)$ and discuss results on its mean-square, we need one more supposition which is most efficiently expressed in geometric language:

Definition. A differentiable function f is said to satisfy the *tangent condition* if none of the tangents to the graph of f contains the origin.

In a first paper on the subject [16], the author established the lower bound

$$\limsup_{t\to\infty} \left(\frac{-\mathrm{sgn}(f'')\mathcal{S}(t)}{t^{1/2}(\log t)^{1/4}}\right) > 0 \tag{1.3}$$

and the mean-square estimate

$$\int_0^T (\mathcal{S}(t))^2 \ \mathrm{d}t \ll T^2 \tag{1.4}$$

as $T \to \infty$, for f satisfying the *tangent condition*. This latter result was subsequently sharpened by Kühleitner and Nowak [13] who obtained the asymptotic formula

$$\int_0^T (\mathcal{S}(t))^2 \ \mathrm{d}t \sim C\,T^2 \tag{1.5}$$

with

$$C = C_{f;a,b} = \frac{1}{4\pi^2} \sum_{\substack{(m_1,h_1),(m_2,h_2)\in\mathcal{D} \\ G(m_1,h_1)=G(m_2,h_2)}}^{*} (h_1h_2)^{-3/2}\kappa(m_1,h_1)\kappa(m_2,h_2) \tag{1.6}$$

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(1) Actually, M. Huxley has meanwhile improved further this upper bound, essentially replacing the exponent $\frac{46}{73} = 0.6301\ldots$ by $\frac{131}{208} = 0.6298\ldots$. The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

(2) Huxley's additional technical condition, that $f^{(3)}$ has no zero on $[a, b]$, actually can be removed, as was shown by G. Kuba (personal communication). In fact, in his book [11], Huxley always avoids it by a suitable change of the coordinate system.

where $(m_1, h_1), (m_2, h_2)$, are elements of $\mathbb{Z} \times \mathbb{N}^*$ in the domain

$$\mathcal{D} = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^+ \,:\, -\eta f'(a) \le \xi \le -\eta f'(b)\ \},$$

$$\kappa(\xi, \eta) = \left| f''\left(\varphi\left(-\frac{\xi}{\eta}\right)\right)\right|^{-1/2}, \quad G(\xi, \eta) = \eta f\left(\varphi\left(-\frac{\xi}{\eta}\right)\right) + \xi\varphi\left(-\frac{\xi}{\eta}\right), \tag{1.7}$$

φ denotes the inverse function of f', and $\sum^*$ means that if (m_1, h_1) lies on the boundary of $\mathcal{D}$, $\kappa(m_1, h_1)$ gets a factor $\frac{1}{2}$, and accordingly for (m_2, h_2). (Here and throughout what follows, notation is adapted to the case that $f'' < 0$ throughout. This is technically convenient but no real loss of generality: If $f'' > 0$, one can replace f by $-f$ and ψ by $-\psi$, with (1.1) still valid.)

Roughly speaking, (1.4) and (1.5) say that

$$\mathcal{S}(t) \ll t^{1/2} \qquad \text{in mean-square.} \tag{1.8}$$

In the present note we investigate the question whether this "average moderate size" of $\mathcal{S}(t)$ can be observed only "in the long run", i.e., by averaging over an interval of order T, or if a similar asymptotic result is true also for a "short interval mean". In fact, it will turn out that (1.8) is true already for an interval of logarithmic length: For any fixed $c_1 > 0$,

$$\int\limits_{T - c_1 \log T}^{T + c_1 \log T} (\mathcal{S}(t))^2 \,\mathrm{d}t \ll T \log T\,. \tag{1.9}$$

As soon as the interval becomes a little longer, we obtain the same asymptotic behaviour as stated in (1.5):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a function with 4 continuous derivatives, which satisfies the tangent condition and $f'' < 0$ throughout. Let further T be a large real parameter and $T \mapsto \Lambda = \Lambda(T)$ an increasing function such that $\Lambda(T) \le \frac{1}{2}T$ and*

$$\lim_{T \to \infty} \frac{\log T}{\Lambda(T)} = 0\,. \tag{1.10}$$

Then, as $T \to \infty$,

$$\int\limits_{T - \Lambda}^{T + \Lambda} (\mathcal{S}(t))^2 \,\mathrm{d}t \sim 4C\Lambda T\,, \tag{1.11}$$

with C as defined in (1.6).

Furthermore, we are able to sharpen the statement of Theorem 1, provided that f is an algebraic function and the limits of summation a, b are algebraic numbers:

Theorem 2. *Let f satisfy the requirements of Theorem 1. Suppose further that a, b are algebraic numbers and that there exists a polynomial $p \in \mathbb{Z}[X, Y]$ such that*

$$p(x, f(x)) = 0$$

for all $x \in [a,b]$. *Assume that, for some* $\delta > 0$,

$$\Lambda = \Lambda(T) \gg T^{\delta}\,, \tag{1.12}$$

as $T \to \infty$. *Then it follows that*

$$\int\limits_{T-\Lambda}^{T+\Lambda} (\mathcal{S}(t))^2 \,\mathrm{d}t = 4C\Lambda T\left(1 + O(T^{-\omega})\right)$$

for some $\omega > 0$ *depending on* f, a, b, *and* δ.

Remarks. 1. The whole type of question, too, has its origin in a paper of HUXLEY's [10] who was the first to consider the corresponding problem for the lattice rest of a convex planar domain (with smooth boundary of finite nonzero curvature throughout), linearly dilated by a large factor t. Catching a word of HUXLEY [10] (who imagined the dilation factor t as a time variable), we can say that, according to our results, these fractional part sums $\mathcal{S}(t)$ "*have no memory*", or, a bit more precisely, that their average small size is accomplished "*not by long-term memory, but by short-term memory*".

2. The author has treated elsewhere [17] the analogous problem for the lattice point discrepancy of the convex disc, obtaining a result similar to Theorem 1. But although this discrepancy can be approximated well by several (!) fractional part sums, there is apparently no simple way to derive these asymptotic results mutually from each other. Furthermore, the methods of proof are (and must be!) quite different: While the argument in [17] uses the two-dimensional Poisson's formula and HLAWKA's [6], [7] asymptotic expansion of the Fourier transform of the indicator function of convex sets, the present paper employs instead Vaaler's approximation (see Lemma 1 below) and a sharp form of the van der Corput transformation.

2. Some auxiliary results.

Lemma 1. (Transition from fractional parts to trigonometric polynomials according to VAALER [18].) *For arbitrary* $w \in \mathbb{R}$ *and* $H \in \mathbb{N}^*$, *suppose that* ψ *satisfies* (1.1), *and put*

$$\psi_H^*(w) = -\frac{1}{\pi}\sum_{h=1}^{H} \frac{\sin(2\pi h w)}{h}\,\tau\left(\frac{h}{H+1}\right),$$

where

$$\tau(\xi) = \pi\xi(1-\xi)\cot(\pi\xi) + \xi \qquad \textit{for } 0 < \xi < 1\,.$$

Then there holds the inequality

$$|\psi(w) - \psi_H^*(w)| \le \frac{1}{H+1}\sum_{h=1}^{H}\left(1 - \frac{h}{H+1}\right)\cos(2\pi h w) \;+\; \frac{1}{2H+2}\,.$$

Proof. For $w \notin \mathbb{Z}$, this is one of the main results in Vaaler [18]. For a very well readable exposition, see also the book of Graham and Kolesnik [5], p. 116. The case $w \in \mathbb{Z}$ is an obvious consequence by a limit argument or by direct evaluation.

Lemma 2. (Analytic properties of the function G.) *Let $G : \mathcal{D} \to \mathbb{R}$ be defined as in* (1.7)*, where f satisfies the tangent condition. Then G is homogeneous of degree 1 and possesses continuous partial derivatives up to order 3 throughout. Furthermore,*

(i) *$G(\xi, \eta)$ is of the same sign on all of $\mathcal{D}$.*

(ii) *There exist constants $c_2 > c_1 > 0$, such that, for all $(\xi, \eta) \in \mathcal{D}$,*

$$c_1\sqrt{\xi^2 + \eta^2} \le |G(\xi, \eta)| \le c_2\sqrt{\xi^2 + \eta^2}\,.$$

(iii) *For Y large and $\Delta > 0$, let $N(Y, \Delta)$ denote the number of (integer) lattice points in*

$$\{(\xi, \eta) \in \mathcal{D} : \; Y - \Delta \le |G(\xi, \eta)| \le Y \;\}\,.$$

Then it follows that

$$N(Y, \Delta) \ll Y^{2/3} + \Delta\, Y\,.$$

Proof. This follows from the discussion in [16], pp. 506, 507, in particular (3.1) and Lemma 4. This last result is based on Branton and Sargos [2].

Lemma 3. (An algebraic property of G.) *Suppose that f satisfies the requirements of Theorem 2, and define $G : \mathcal{D} \to \mathbb{R}$ again by* (1.7)*. Then, for all $(m_1, h_1), (m_2, h_2) \in \mathbb{Z}^2 \cap \mathcal{D}$ with $G(m_1, h_1) \ne G(m_2, h_2)$,*

$$|G(m_1, h_1) - G(m_2, h_2)| \gg \left(\max\{|m_1|\,, h_1, |m_2|\,, h_2\}\right)^{-A}\,,$$

where $A > 0$ depends only on f.

Proof. This is clause (ii) of Lemma 4.1 in [13].

3. Proof of Theorem 1. Throughout the sequel, let T and M be large real numbers, with $T \ge M^2$, but otherwise independent of each other. All O- and $\ll$-constants may depend on f and $[a, b]$, but not on T and M. Suppose further that $t \in [T - \Lambda, T + \Lambda] \subseteq [\frac{1}{2}T, \frac{3}{2}T]$, thus $t \asymp T$ as $T \to \infty$.

For any complex-valued function $\Phi : t \mapsto \Phi(t)$ which is square- integrable on $[T - \Lambda, T + \Lambda]$, we define

$$\mathcal{M}_T(\Phi) = \int_{T-\Lambda}^{T+\Lambda} |\Phi(t)|^2 \,\mathrm{d}t\,. \tag{3.1}$$

By Cauchy's inequality, for arbitrary $\Phi_1, \Phi_2 \in L^2[T - \Lambda, T + \Lambda]$,

$$\mathcal{M}_T(\Phi_1 + \Phi_2) \le 2\left(\mathcal{M}_T(\Phi_1) + \mathcal{M}_T(\Phi_2)\right)\,, \tag{3.2}$$

which will be used frequently in what follows.

By means of Lemma 1, we shall approximate $\mathcal{S}(t)$ by

$$\mathcal{S}_H^*(t) = \sum_{at<n\le bt,\ n\in\mathbb{Z}} \psi_H^*\left(tf\left(\frac{n}{t}\right)\right), \tag{3.3}$$

with $H=[T]$. By the definition of ψ_H^*,

$$\mathcal{S}_H^*(t) = \sum_{1\le h\le H} \beta(h,H) \sum_{at<n\le bt} \sin\left(-2\pi htf\left(\frac{n}{t}\right)\right), \tag{3.4}$$

where $\beta(h,H)=\frac{1}{\pi}h^{-1}\tau\left(\frac{h}{H+1}\right)$. The innermost sum on the right hand side is subject to a sharp form of van der Corput transformation ("B-step"). See [13], Lemmas 2.2 and 2.4, for details. In particular, we use formula (3.2) of [13] which reads (with $e(z)=e^{2\pi iz}$ as usual)

$$\begin{aligned}\sum_{at<n\le bt} e\left(-htf\left(\frac{n}{t}\right)\right) = \frac{\sqrt{t}}{\sqrt{h}} \sideset{}{''}\sum_{-hf'(a)\le m\le -hf'(b)} \kappa(m,h) e\left(-tG(m,h)+\tfrac{1}{8}\right) + \\ + O(\log(1+h)) + O(r_h(a)) + O(r_h(b)),\end{aligned} \tag{3.5}$$

where

$$r_h(c) = \begin{cases} 0 & \text{if } hf'(c)\in\mathbb{Z}, \\ \min\left(\|hf'(c)\|^{-1}, \frac{\sqrt{T}}{\sqrt{h}}\right) & \text{else,}\end{cases} \qquad (c=a \text{ or } b)$$

$\|\cdot\|$ denoting the distance from the nearest integer, and

$$\sideset{}{''}\sum_{a\le n\le b} \Phi(n) := \frac{\chi_{\mathbb{Z}}(a)\Phi(a)+\chi_{\mathbb{Z}}(b)\Phi(b)}{2} + \sum_{a<n<b} \Phi(n),$$

$\chi_{\mathbb{Z}}(\cdot)$ the indicator function of the integers. We estimate the contribution of the error terms in (3.5) to $\mathcal{M}_T(S_H^*)$: If $f'(a)\in\mathbb{Q}$, $r_h(a)$ is bounded. Therefore, since $\beta(h,H)\ll h^{-1}$,

$$\sum_{1\le h\le H} \beta(h,H) r_h(a) \ll \sum_{1\le h\le H} h^{-1} \ll \log H.$$

If $f'(a)\notin\mathbb{Q}$,

$$\sum_{1\le h\le M} \beta(h,H) r_h(a) + \sum_{M<h\le H} \beta(h,H) r_h(a) \ll$$

$$\ll \log M \max_{1\le h\le M} \|hf'(a)\|^{-1} + \sqrt{T}\sum_{h>M} h^{-3/2} \ll K_1(M) + M^{-1/2}T^{1/2}.$$

Here and in what follows, $K_1(M), K_2(M), \ldots$ are suitable positive numbers depending on M but not on T. Since the same argument applies to $r_h(b)$, we conclude that

$$\mathcal{M}_T\left(\sum_{1\le h\le H} \beta(h,H)\left(O(\log(1+h)) + O(r_h(a)) + O(r_h(b))\right)\right) \ll$$

$$\ll \Lambda \left((\log T)^4 + (K_1(M))^2 + M^{-1} T \right) . \tag{3.6}$$

Using the imaginary part of (3.5) to transform the inner sums in (3.4), apart from error terms[(3)], we arrive at

$$\mathcal{S}_H^{**}(t) := -\sqrt{t} \sum_{(m,h)\in\mathcal{D}^{(H)}}^{*} \frac{\beta(h,H)}{\sqrt{h}} \kappa(m,h) \sin(2\pi t G(m,h) - \tfrac{\pi}{4}) ,$$

where

$$\mathcal{D}^{(H)} := \{ (m,h) \in \mathbb{Z}^2 \ : \ -hf'(a) \le m \le -hf'(b), \ 1 \le h \le H \ \} .$$

Actually, the main contribution to our mean-square asymptotics will come from a truncation[(4)] of this sum, namely from

$$\Sigma_M(t) := -\sqrt{t} \sum_{\substack{(m,h)\in\mathcal{D}, \\ |G(m,h)|\le M}}^{*} \frac{\beta(h,H)}{\sqrt{h}} \kappa(m,h) \sin(2\pi t G(m,h) - \tfrac{\pi}{4}) . \tag{3.7}$$

Evidently,

$$\mathcal{S}_H^{**}(t) - \Sigma_M(t) \ll$$

$$\ll T^{1/2} \left| \sum_{(m,h)\in\mathcal{D}^{(H)}}^{*} \frac{\gamma_1(m,h,H)}{\sqrt{h}} \kappa(m,h) e(tG(m,h)) \right| , \tag{3.8}$$

with

$$\gamma_1(m,h,H) := \begin{cases} \beta(h,H) = \frac{1}{\pi} h^{-1} \tau\left(\frac{h}{H+1}\right) & \text{if } |G(m,h)| > M, \\ 0 & \text{else.} \end{cases} \tag{3.9}$$

Further, by Lemma 1,

$$\mathcal{S}(t) - \mathcal{S}_H^{*}(t) = \sum_{at<n\le bt, \ n\in\mathbb{Z}} \left(\psi\left(tf\left(\frac{n}{t}\right)\right) - \psi_H^{*}\left(tf\left(\frac{n}{t}\right)\right)\right) \ll$$

$$\ll \sum_{1\le h\le H} \frac{1-\frac{h}{H+1}}{H+1} \sum_{at<n\le bt} \cos\left(2\pi h t f\left(\frac{n}{t}\right)\right) + \ 1 .$$

Applying (3.5) to this sum over cosines, we conclude that this is

$$\ll T^{1/2} \left| \sum_{(m,h)\in\mathcal{D}^{(H)}}^{*} \frac{\gamma_2(m,h,H)}{\sqrt{h}} \kappa(m,h) e(tG(m,h)) \right| \ + \ \rho_T(t) , \tag{3.10}$$

where

$$\gamma_2(m,h,H) := \frac{1-\frac{h}{H+1}}{H+1} \tag{3.11}$$

(3) Note that $\mathcal{M}_T\left(\mathcal{S}_H^{*} - \mathcal{S}_H^{**}\right)$ has already been estimated by (3.6).

(4) Because of $T \ge M^2$, one can replace $\mathcal{D}^{(H)}$ by $\mathcal{D}$ in the summation condition of $\Sigma_M(t)$, still getting a proper subsum of $\mathcal{S}_H^{**}(t)$.

and $\rho_T(t)$ is an error term which, by a repetition of the estimate which lead to (3.6), satisfies

$$\mathcal{M}_T\left(\rho_T\right) \ll \Lambda\left((\log T)^4 + (K_1(M))^2 + M^{-1}T\right) . \tag{3.12}$$

We observe that the right hand sides of (3.8) and (3.10) are fairly similar. Therefore, their mean-squares can be estimated by basically the same calculation. We put

$$\mathcal{R}_j(t) := \sum_{(m,h)\in\mathcal{D}^{(H)}}^{*} \frac{\gamma_j(m,h,H)}{\sqrt{h}}\kappa(m,h)e(tG(m,h))$$

where γ_j is either of γ_1, γ_2. In order to bound $\mathcal{M}_T\left(\mathcal{R}_j\right)$, we employ a device used extensively by Huxley [10] which involves the Fejér kernel

$$\phi(w) := \left(\frac{\sin(\pi w)}{\pi w}\right)^2 .$$

By Jordan's inequality, $\phi(w) \geq \frac{4}{\pi^2}$ for $|w| \leq \frac{1}{2}$, and the Fourier transform is simply

$$\widehat{\phi}(z) = \int_{\mathbb{R}} \phi(w)e(wz)\,\mathrm{d}w = \max(0, 1-|z|)\,.$$

Consequently,

$$\mathcal{M}_T\left(\mathcal{R}_j\right) = 2\Lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\mathcal{R}_j(T+2\Lambda w)\right|^2\,\mathrm{d}w \leq \frac{\pi^2}{2}\Lambda \int_{\mathbb{R}} \phi(w)\left|\mathcal{R}_j(T+2\Lambda w)\right|^2\,\mathrm{d}w =$$

$$= \frac{\pi^2}{2}\Lambda \sum_{\substack{(m_1,h_1)\in\mathcal{D}^{(H)},\\ (m_2,h_2)\in\mathcal{D}^{(H)}}}^{*} \frac{\gamma_j(m_1,h_1,H)\gamma_j(m_2,h_2,H)}{\sqrt{h_1h_2}}\kappa(m_1,h_1)\kappa(m_2,h_2)\times$$

$$\times e\left(T(G(m_1,h_1)-G(m_2,h_2))\right)\int_{\mathbb{R}} \phi(w)e\left(2\Lambda w\left(G(m_1,h_1)-G(m_2,h_2)\right)\right)\mathrm{d}w \ll$$

$$\ll \Lambda \sum_{\substack{(m_1,h_1)\in\mathcal{D}^{(H)},\\ (m_2,h_2)\in\mathcal{D}^{(H)}}}^{*} \frac{\gamma_j(m_1,h_1,H)\gamma_j(m_2,h_2,H)}{\sqrt{h_1h_2}} \max\left(0, 1-2\Lambda\left|G(m_1,h_1)-G(m_2,h_2)\right|\right) , \tag{3.13}$$

where we have used that $\kappa(m,h) \ll 1$. We recall that $(m,h)\in\mathcal{D}$ implies that $m \ll h$ and hence, in view of Lemma 2, (ii), that

$$|G(m,h)| \asymp \sqrt{m^2+h^2} \asymp h\,. \tag{3.14}$$

Further, if a term of the last sum is nonzero, necessarily

$$\left||G(m_1,h_1)| - |G(m_2,h_2)|\right| < (2\Lambda)^{-1}\,,$$

hence $h_1 \asymp h_2$. Thus we infer from (3.13) that

$$\mathcal{M}_T\left(\mathcal{R}_j\right) \ll \Lambda \sum_{(m_1,h_1)\in\mathcal{D}^{(H)}} \frac{\gamma_j(m_1,h_1,H)}{h_1} \underbrace{\sum_{\substack{(m_2,h_2)\in\mathcal{D}^{(H)}:\\ ||G(m_1,h_1)|-|G(m_2,h_2)||<(2\Lambda)^{-1}}} \gamma_j(m_2,h_2,H)}_{S_j(m_1,h_1)} .$$

Now we have to distinguish if $j=1$ or 2, recalling the respective definitions (3.9) and (3.11) of the γ-coefficients. For $j=1$, it follows that $S_1(m_1,h_1)=0$ if $|G(m_1,h_1)| > M-1$, and otherwise

$$S_1(m_1,h_1) \ll \frac{1}{h_1}\,\#\{(m,h)\in\mathcal{D}^{(H)}:\ ||G(m,h)|-|G(m_1,h_1)|| < (2\Lambda)^{-1}\} \ll$$

$$\ll \frac{1}{h_1}\left(|G(m_1,h_1)|^{2/3} + \frac{1}{\Lambda}\,|G(m_1,h_1)|\right) \ll h_1^{-1/3} + \frac{1}{\Lambda} .$$

Here we have used Lemma 2, clause (iii). Thus altogether

$$\begin{aligned} \mathcal{M}_T\left(\mathcal{R}_1\right) &\ll \Lambda \sum_{\substack{(m_1,h_1)\in\mathcal{D}^{(H)}:\\ h_1\gg M}} h_1^{-2}\left(h_1^{-1/3} + \frac{1}{\Lambda}\right) \ll \\ &\ll \Lambda \sum_{h_1\gg M} h_1^{-4/3} + \sum_{1\le h_1\ll T} h_1^{-1} \ll \Lambda M^{-1/3} + \log T . \end{aligned} \tag{3.15}$$

Dealing with the case $j=2$, we can use that $\gamma_2(m,h,H) \ll H^{-1}$. Therefore, by the same argument,

$$\begin{aligned} \mathcal{M}_T\left(\mathcal{R}_2\right) &\ll \frac{\Lambda}{T^2} \sum_{(m_1,h_1)\in\mathcal{D}^{(H)}} \left(h_1^{-1/3} + \frac{1}{\Lambda}\right) \ll \\ &\ll \frac{1}{T^2} \sum_{1\le h_1\ll T} \left(\Lambda h_1^{2/3} + h_1\right) \ll \Lambda T^{-1/3} + 1 . \end{aligned} \tag{3.16}$$

What we have proved so far can be summarized as follows:

Proposition. *Uniformly in* $t\in[T-\Lambda, T+\Lambda]$,

$$\mathcal{S}(t) = \Sigma_M(t) + \Delta_M(t) ,$$

where

$$\mathcal{M}_T\left(\Delta_M\right) \ll \Lambda (K_1(M))^2 + T\left(\Lambda M^{-1/3} + \log T\right)$$

and $\Sigma_M(t)$ *has been defined in* (3.7).

By the definition in Lemma 1, $\tau(w) = 1 + O(w^2)$, hence

$$\beta(h,H) = \frac{1}{h\pi}\tau\left(\frac{h}{H+1}\right) = \frac{1}{h\pi} + O\left(\frac{h}{H^2}\right).$$

Consequently, if we put

$$\Sigma_M^{(0)}(t) := -\sqrt{t} \sum_{\substack{(m,h)\in\mathcal{D},\\ |G(m,h)|\le M}}^{*} \frac{1}{\pi\, h^{3/2}}\kappa(m,h)\sin(2\pi t G(m,h) - \tfrac{\pi}{4})$$

it is easy to see that

$$\Sigma_M(t) = \Sigma_M^{(0)}(t) + O\left(K_2(M)T^{-3/2}\right). \tag{3.17}$$

In order to evaluate $\mathcal{M}_T\left(\Sigma_M^{(0)}\right)$, we square out $(\Sigma_M^{(0)}(t))^2$, using the elementary formula

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right),$$

and integrate over $t \in [T-\Lambda, T+\Lambda]$. The main contribution comes from the diagonal terms, i.e., those with $G(m_1,h_1) = G(m_2,h_2)$ (and the $\cos(A-B)$-part of the above formula), and reads

$$\frac{\Lambda T}{\pi^2} \sum_{\substack{(m_1,h_1),(m_2,h_2)\in\mathcal{D}\\ |G(m_1,h_1)|=|G(m_2,h_2)|\le M}}^{*} \frac{\kappa(m_1,h_1)\kappa(m_2,h_2)}{(h_1h_2)^{3/2}}.$$

Extending the range of summation to all of $\mathcal{D}$ just gives $4C\Lambda T$, the main term in our Theorems. To estimate the error caused by this extension, we appeal once more to Lemma 2, (iii), to conclude that

$$\sum_{\substack{(m,h)\in\mathcal{D}\\ |G(m,h)|=X}} 1 \ll X^{2/3}$$

for X large. Therefore, recalling (3.14) and the fact that $\kappa(m,h) \ll 1$,

$$\sum_{\substack{(m_1,h_1),(m_2,h_2)\in\mathcal{D}\\ |G(m_1,h_1)|=|G(m_2,h_2)|>M}}^{*} \frac{\kappa(m_1,h_1)\kappa(m_2,h_2)}{(h_1h_2)^{3/2}} \ll$$

$$\ll \sum_{\substack{(m_1,h_1)\in\mathcal{D}\\ |G(m_1,h_1)|>M}} |G(m_1,h_1)|^{-3} \sum_{\substack{(m_2,h_2)\in\mathcal{D}\\ |G(m_2,h_2)|=|G(m_1,h_1)|}} 1 \ll$$

$$\ll \sum_{\substack{(m_1,h_1)\in\mathcal{D}\\ |G(m_1,h_1)|>M}} |G(m_1,h_1)|^{-7/3} \ll \sum_{h\gg M} h^{-4/3} \ll M^{-1/3}.$$

All other terms arising in the evaluation of $\mathcal{M}_T\left(\Sigma_M^{(0)}\right)$ are rather small: Actually,

$$\int\limits_{T-\Lambda}^{T+\Lambda} \begin{matrix}\cos\\ \sin\end{matrix} \left(2\pi t\left(G(m_1,h_1)\pm G(m_2,h_2)\right)\right) t \,\mathrm{d}t \ll \frac{T}{|G(m_1,h_1)\pm G(m_2,h_2)|}\,, \tag{3.18}$$

which contributes overall only $\ll K_3(M)T$. Hence,

$$\mathcal{M}_T\left(\Sigma_M^{(0)}\right) = 4C\Lambda T + O\left(\Lambda T M^{-1/3}\right) + O\left(K_3(M)T\right)\,.$$

Combining this with (3.17) and our Proposition, and appealing to (3.2) and to Cauchy's inequality, we arrive at

$$\begin{aligned}\mathcal{M}_T\left(\mathcal{S}\right) = 4C\Lambda T &+ O\left(\Lambda T^{1/2}(K_1(M)^2 + K_2(M)^2)\right) + O\left(\Lambda T M^{-1/6}\right)\\ &+ O\left(T(\Lambda\log T)^{1/2}\right) + O\left(K_3(M)T\right)\,.\end{aligned} \tag{3.19}$$

Therefore, for any fixed M,

$$\limsup_{T\to\infty}\left|\frac{1}{\Lambda T}\mathcal{M}_T\left(\mathcal{S}\right) - 4C\right| \ll M^{-1/6}\,,$$

by the condition (1.10). Since M can be chosen arbitrarily large, the proof of Theorem 1 is thereby complete.

To establish (1.9), we choose for M any fixed value, and $\Lambda = c_1\log T$, then (3.19) proves what we want.

4. Proof of Theorem 2. Our task is to replace $K_1(M), K_2(M), K_3(M)$ in the above argument by explicit powers of T, under the stronger conditions of Theorem 2. (Cf. also section 4 of [13].) We put throughout $M = T^{\varepsilon}$, with some suitably small $\varepsilon > 0$. First we reconsider the estimate which lead to (3.6). For $f'(a)$ rational, no change is necessary. If $f'(a)$ is irrational, it must be an algebraic number. We conclude that

$$\begin{aligned}&\sum_{1\le h\le T^{\varepsilon}}\beta(h,H)r_h(a) + \sum_{T^{\varepsilon}<h\le H}\beta(h,H)r_h(a) \ll\\ &\ll \sum_{1\le h\le T^{\varepsilon}}\frac{1}{h\,\|hf'(a)\|} + \sqrt{T}\sum_{h>T^{\varepsilon}} h^{-3/2} \ll T^{C_1\varepsilon} + T^{(1-\varepsilon)/2}\,.\end{aligned} \tag{4.1}$$

Here we have used that

$$\sum_{1\le h\le W}\frac{1}{h\,\|h\alpha\|} \ll W^{N-1}\log W$$

for α any algebraic number of degree $N\ge 2$. (Cf. Lemma 4.2 of [13].) Hence we obtain, instead of (3.6),

$$\mathcal{M}_T\left(\mathcal{S}_H^* - \mathcal{S}_H^{**}\right) \ll \Lambda\left((\log T)^4 + T^{2C_1\varepsilon} + T^{(1-\varepsilon)}\right)\,. \tag{4.2}$$

Next we consider (3.17). The estimate concerned now reads

$$\sqrt{t}\sum_{(m,h)\in\mathcal{D},\ |G(m,h)|\le T^{\varepsilon}}^{*}\frac{h^{1/2}}{T^2}\kappa(m,h)\ll T^{-3/2}\sum_{1\le h\ll T^{\varepsilon}}h^{3/2}\ll 1$$

which contributes only $O(\Lambda)$ to the mean-square integral. Finally, returning to (3.18), we appeal to Lemma 3 to conclude that

$$\sum_{\mathrm{SC}}\frac{(h_1h_2)^{-3/2}\,T}{|G(m_1,h_1)-G(m_2,h_2)|}\ll$$

$$\ll T^{1+A\varepsilon}\sum_{h_1,h_2\ll T^{\varepsilon}}(h_1h_2)^{-1/2}\ll T^{1+(A+1)\varepsilon}\,,$$

with the summation condition

$$\mathrm{SC}:(m_1,h_1),(m_2,h_2)\in\mathcal{D},\ G(m_1,h_1)\ne G(m_2,h_2),\ |G(m_1,h_1)|\,,|G(m_2,h_2)|\le T^{\varepsilon}\,.$$

The same bound applies *a fortiori* to the sum involving $|G(m_1,h_1)+G(m_2,h_2)|$. Hence, altogether, (3.19) is now replaced by

$$\begin{aligned}\mathcal{M}_T(\mathcal{S})=4C\Lambda T&+O\left(\Lambda(T^{\frac{1}{2}+C_1\varepsilon}+T^{1-\varepsilon/2})\right)+O\left(\Lambda T^{1-\varepsilon/6}\right)\\&+O\left(T(\Lambda\log T)^{1/2}\right)+O\left(T^{1+(A+1)\varepsilon}\right)\,.\end{aligned}\tag{4.3}$$

In view of condition (1.12), this readily implies (for ε sufficiently small)

$$\mathcal{M}_T(\mathcal{S})=4C\Lambda T\left(1+O\left(T^{-\omega}\right)\right)\qquad(\omega>0)\,,$$

the assertion of Theorem 2.

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