Diophantine approximation in $\mathbb{R}^s$: On a method of Mordell and Armitage

Werner Georg NOWAK

Abstract. In this article we establish an inequality involving the critical determinants of star bodies in unequal dimensions, in the spirit of a method due to Mordell and Armitage from the "Golden Age" of the Geometry of Numbers. Applying this repeatedly and using the value of the critical determinant of a (three-dimensional) double paraboloid determined just recently [14], we derive a bound for the Diophantine approximation constant in $\mathbb{R}^5$ with respect to the Euclidean norm.

AMS-Classification: 11H06, 11J13

1. Introduction. θ_s, the s-dimensional simultaneous Diophantine approximation constant (with respect to the Euclidean norm $\|\cdot\|_2$) is defined as the supremum of all reals c with the property that, for any s-tuple $\mathbf{x} \in \mathbb{R}^s - \mathbb{Q}^s$, there exist infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{N}^*$ satisfying

$$\left\| \mathbf{x} - \frac{1}{q}\mathbf{p} \right\|_2 < \frac{1}{c^{1/s} q^{1+1/s}} . \tag{1.1}$$

By Hurwitz' classic theorem, $\theta_1 = \sqrt{5}$. According to a deep result of Davenport and Mahler [5], $\theta_2 = \frac{1}{2}\sqrt{23}$. For all other s, the exact values of θ_s are unknown with only more or less precise bounds available.$^{(1)}$
By a celebrated theorem of Davenport [4], θ_s is equal to the critical determinant$^{(2)}$ of the $(s+1)$-dimensional star body

$$K_{s+1} : \quad |x_0| \left(x_1^2 + \ldots + x_s^2 \right)^{s/2} \le 1 . \tag{1.2}$$

Prasad [16] used this fact and the obvious idea to inscribe an ellipsoid into K_{s+1} by the arithmetic-geometric mean inequality. Thus he deduced that

$$\theta_s = \Delta(K_{s+1}) \ge \frac{(s+1)^{(s+1)/2}}{s^{s/2}} \, \Delta(\mathcal{S}_{s+1}) , \tag{1.3}$$

---

(1) We remark parenthetically that the problem becomes still harder if $\|\cdot\|_2$ is replaced by $\|\cdot\|_\infty$. Here, according to Mack [8], the author [12], and Cassels [3], $\left(\frac{13}{8}\right)^2 \le \theta_2^{(\infty)} \le \frac{7}{2}$, while for $s \ge 3$, only the approach on the basis of Blichfeldt's [2] classic theorem has been carried out successfully; this was given its sharpest possible form by Spohn [17].

(2) Let us recall briefly some basic concepts from the Geometry of Numbers: A lattice $\Lambda = \mathbf{A}\mathbb{Z}^s$ ($\mathbf{A}$ a real non-singular $(s \times s)$-matrix) is called *admissible* for a star body K if the only lattice point of Λ contained in the interior of K is the origin $\mathbf{o}$. Further, the *critical determinant* $\Delta(K)$ of the body K is defined as the infimum of all lattice constants $d(\Lambda) = |\det \mathbf{A}|$ where Λ ranges over all lattices admissible for K. (Cf. throughout Gruber/Lekkerkerker [6].)

$\mathcal{S}_{s+1}$ the $(s+1)$-dimensional unit sphere. For $s \geq 4$, this was essentially the sharpest bound available until recently (apart from a numerically insignificant refinement due to the author [13]).

For $s = 3$, an ingenious different approach was brought to a success by ARMITAGE [1]: On the basis of a method of MORDELL [9], [10], [11], he proved a general theorem which included the assertion that $\Delta(K_{s+1})$ is related to the critical determinant of the s-dimensional star body

$$K^*_{s,s} \ : \quad |x_1| \left(x_1^2 + \ldots + x_s^2\right)^{s/2} \leq 1 \tag{1.4}$$

by the inequality

$$\Delta(K_{s+1}) \geq \Delta(K^*_{s,s})^{\frac{s+1}{s-1}} \, . \tag{1.5}$$

Furthermore, he was able to estimate $\Delta(K^*_{3,3})$ in terms of the critical determinants of a planar domain and some other three-dimensional body. (This can be viewed as the special case $s = p = 3$ of our Theorem 1 below.) In this way he obtained

$$3.1914\ldots = \sqrt{\frac{275}{27}} \ \geq \theta_3 \geq \ 1.159^3 \, \tfrac{1}{2} \, 3^{3/4} = 1.774\ldots \tag{1.6}$$

However, ARMITAGE's arguments turned out to be not entirely complete, as was pointed out by subsequent authors (see, e.g., LEKKERKERKER [7]). But everything can be saved, and this is done so for the special purpose needed here by our Lemma 1 below.

2. The Main Theorem. The objective of the present article is to establish – by means of the Mordell-Armitage method – a general inequality involving the critical determinants of related star bodies in unequal dimensions. This will be suitable for iterated application (starting from $\Delta(K^*_{s,s})$), in order to arrive finally at a star body of low dimension whose critical determinant can be estimated well.

Preliminaries(3)**.** If $\Lambda = \mathbf{A}\mathbb{Z}^s$ is a lattice in $\mathbb{R}^s$, the lattice Λ^* generated by ${}^t\mathbf{A}^{-1}$ (the transposed of the inverse of $\mathbf{A}$) is called the *polar* lattice of Λ. Evidently, for any points $\mathbf{x} = \mathbf{A}\mathbf{u} \in \Lambda$, $\mathbf{y} = {}^t\mathbf{A}^{-1}\,\mathbf{v} \in \Lambda^*$ $(\mathbf{u}, \mathbf{v} \in \mathbb{Z}^s)$, it follows that $\mathbf{x}\cdot\mathbf{y} = \mathbf{u}\cdot\mathbf{v} \in \mathbb{Z}$.

Lemma 1. *If a lattice Λ does not contain any nontrivial point of the form $(0, x_2, \ldots, x_s)$ then Λ^* contains no nontrivial point of the form $(y_1, 0, \ldots, 0)$.*

Proof. Suppose that $\mathbf{y} = (y_1, 0, \ldots, 0) \in \Lambda^*$, $y_1 \neq 0$. Then, for all nontrivial points $\mathbf{x} = (x_1, \ldots, x_s)$ of Λ, $\mathbf{x}\cdot\mathbf{y} = x_1 y_1$ is a nonzero integer (since $x_1 \neq 0$). Thus

$$|x_1| \geq \frac{1}{|y_1|} \tag{$*$}$$

(3) Points and vectors are always meant as *column vectors* although we write them – for convenience of print – as s-tuples in one line.

for all nontrivial points $\mathbf{x} = (x_1, \ldots, x_s)$ of Λ. Now consider the convex body

$$K = \{(t_1, \ldots, t_s) \in \mathbb{R}^s : \ |t_1| \le \frac{1}{2\,|y_1|}, \ |t_j| \le (2d(\Lambda)\,|y_1|)^{1/(s-1)} \ \ (j = 2, \ldots, s) \ \}.$$

Its volume is $2^s d(\Lambda)$, hence Minkowski's Convex Body Theorem yields a contradiction against $(*)$.

We are now ready to state our main theorem.

Theorem 1. *For arbitrary positive integers $s \ge 3$ and p, define the s-dimensional star bodies $K^*_{s,p}$, $K^{**}_{s,p}$ by*

$$K^*_{s,p} : \qquad x_1^2(x_1^2 + \ldots + x_s^2)^p \le 1\,,$$
$$K^{**}_{s,p} : \qquad (x_1^2 + \ldots + x_s^2)^{p-s+2}(x_2^2 + \ldots + x_s^2)^{s-1} \le 1\,,$$

respectively. Then the inequality

$$\Delta(K^*_{s,p}) \ge \Delta(K^*_{s-1,p})^{\frac{s}{s-1}}\,\Delta(K^{**}_{s,p})^{\frac{1}{s-1}}$$

holds true.

Proof. Let Λ be a lattice in $\mathbb{R}^s$ which is admissible for $K^*_{s,p}$. We put $\alpha = \frac{p-s+2}{2(p+1)}$, $\beta = \frac{s-1}{2(p+1)}$, and consider the star body

$$K^{**}_{s,p}(\lambda) : \qquad (x_1^2 + \ldots + x_s^2)^{\alpha}(x_2^2 + \ldots + x_s^2)^{\beta} \le \lambda$$

where λ is a positive real parameter. Since the left-hand side is homogeneous of degree 1, it follows that $\Delta(K^{**}_{s,p}(\lambda)) = \lambda^s \Delta(K^{**}_{s,p})$. Now pick λ so large that

$$\lambda^s \Delta(K^{**}_{s,p}) > \frac{1}{d(\Lambda)} = d(\Lambda^*)\,. \tag{2.1}$$

It follows that Λ^* is not admissible for $K^{**}_{s,p}(\lambda)$, hence there exists some (nontrivial) primitive lattice point $\mathbf{g} = (g_1, \ldots, g_s)$ of Λ^* such that

$$(g_1^2 + \ldots + g_s^2)^{\alpha}(g_2^2 + \ldots + g_s^2)^{\beta} \le \lambda\,. \tag{2.2}$$

Let $\delta = (g_2^2 + \ldots + g_s^2)^{1/2}$, then $\delta > 0$ by Lemma 1. (Since Λ is admissible for $K^*_{s,p}$, it cannot contain a nontrivial point with first coordinate 0.) The last inequality can thus be written as

$$(g_1^2 + \delta^2)^{\alpha}\delta^{2\beta} \le \lambda\,. \tag{2.3}$$

Further, there obviously exists a rotation ρ which leaves $K^{**}_{s,p}$ invariant and sends $\mathbf{g}$ to $\mathbf{g}^{(\rho)} = (g_1, \delta, 0, \ldots, 0)$. (The restriction of ρ to points of the form $(y_1, 0, \ldots, 0)$ is the identity.) Let $\Lambda_\rho\,, \Lambda^*_\rho$ be the images of Λ, Λ^* under ρ. There exists a basis $\mathbf{B}^*$ of

Λ_ρ^* whose first column is $\mathbf{g}^{(\rho)}$. Then ${}^t\mathbf{B}^{*-1} = \mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_s)$ is a basis of Λ_ρ. For any point $\ell \in \Lambda_\rho$, we may thus write $\ell = (l_1, \ldots, l_s) = \mathbf{B}\mathbf{u}$, $\mathbf{u} = (u_1, \ldots, u_s) \in \mathbb{Z}^s$, and conclude that $\ell \cdot \mathbf{g}^{(\rho)} = l_1 g_1 + l_2 \delta = u_1$. Consequently, if $\varepsilon(u_1)$ denotes the $(s-1)$-dimensional hyperplane (in $(x_1, \ldots, x_s)$-space)

$$\varepsilon(u_1): \quad x_1 g_1 + x_2 \delta = u_1 ,$$

it is clear that every point of Λ_ρ lies in one of the $\varepsilon(u_1)$'s with u_1 ranging over all integers. We now construct an $(s-1)$-dimensional lattice $\hat{\Lambda}$ by projecting the sub-lattice of Λ_ρ contained in $\varepsilon(0)$ into the $(s-1)$-dimensional coordinate hyperplane $\varepsilon_0 \ : \ x_2 = 0$. Let us calculate $d(\hat{\Lambda})$: The normal distance of the hyperplanes $\varepsilon(0)$, $\varepsilon(1)$ is $(g_1^2 + \delta^2)^{-1/2}$, hence $d(\Lambda) = d(\Lambda_\rho) = (g_1^2 + \delta^2)^{-1/2} \mathrm{vol}(\#(\mathbf{b}_2, \ldots, \mathbf{b}_s))$, where $\mathrm{vol}(\#(\mathbf{b}_2, \ldots, \mathbf{b}_s))$ denotes the volume of the fundamental parallelogram spanned by $\mathbf{b}_2, \ldots, \mathbf{b}_s$. (Note that $\mathbf{b}_2, \ldots, \mathbf{b}_s$ are coplanar to $\varepsilon(0)$.) The normal vectors of $\varepsilon(0), \varepsilon_0$ are $\mathbf{g}^{(\rho)}$, $(0, 1, 0, \ldots, 0)$, respectively, hence the normal projection of $\varepsilon(0)$ onto ε_0 reduces $\mathrm{vol}(\#(\mathbf{b}_2, \ldots, \mathbf{b}_s))$ by a factor $\delta(g_1^2 + \delta^2)^{-1/2}$. Thus altogether

$$d(\hat{\Lambda}) = \delta \, d(\Lambda_\rho) . \tag{2.4}$$

Submitting $K_{s,p}^*$ to the same procedure which just created $\hat{\Lambda}$ from $\Lambda^{(\rho)}$, we obtain the $(s-1)$-dimensional star body (in coordinates $(x_1, x_3, \ldots, x_s)$)

$$\mathcal{R}: \qquad x_1^2 \left(x_1^2 + \frac{g_1^2}{\delta^2} x_1^2 + x_3^2 + \ldots + x_s^2 \right)^p \le 1 .$$

By construction, $\hat{\Lambda}$ is admissible for $\mathcal{R}$, hence

$$d(\hat{\Lambda}) \ge \Delta(\mathcal{R}) . \tag{2.5}$$

The substitution

$$x_1 \to x_1 \left(1 + \frac{g_1^2}{\delta^2} \right)^{\frac{p}{2(p+1)}} , \quad x_j \to x_j \left(1 + \frac{g_1^2}{\delta^2} \right)^{-\frac{1}{2(p+1)}} \quad (j = 3, \ldots, s)$$

transforms $\mathcal{R}$ to $K_{s-1,p}^*$. Its determinant is equal to $\left(1 + \frac{g_1^2}{\delta^2} \right)^{\frac{p-s+2}{2(p+1)}}$, thus

$$\Delta(\mathcal{R}) = \left(1 + \frac{g_1^2}{\delta^2} \right)^{-\alpha} \Delta(K_{s-1,p}^*) ,$$

with $\alpha = \frac{p-s+2}{2(p+1)}$ as defined earlier. Combining this with (2.4), (2.5), and (2.3), we get

$$d(\Lambda) \ge (g_1^2 + \delta^2)^{-\alpha} \delta^{-2\beta} \Delta(K_{s-1,p}^*) \ge \frac{1}{\lambda} \Delta(K_{s-1,p}^*) .$$

By construction, we can choose $d(\Lambda)$ arbitrarily close to $\Delta(K^*_{s,p})$ and λ close to $\big(\Delta(K^{**}_{s,p})d(\Lambda)\big)^{-1/s}$ (in view of (2.1)). Therefore,

$$\Delta(K^*_{s,p}) \geq \big(\Delta(K^{**}_{s,p})\Delta(K^*_{s,p})\big)^{1/s}\,\Delta(K^*_{s-1,p})\,,$$

which readily completes the proof of Theorem 1.

3. Application to simultaneous Diophantine approximation. Armitage who had the special case $s = p$ of Theorem 1 already in hands applied this along with (1.5) for $s = p = 3$ to conclude that

$$\theta_3 = \Delta(K_4) \geq \Delta(K^*_{3,3})^2 \geq \Delta(K^*_{2,3})^3\Delta(K^{**}_{3,3})\,.$$

Using a special argument suitable for planar problems to estimate $\Delta(K^*_{2,3})$ and inscribing an ellipsoid into $K^{**}_{3,3}$ he thus obtained

$$\theta_3 = \Delta(K_4) \geq 1.159^3\,\tfrac{1}{2}3^{3/4} = 1.774\ldots \tag{3.1}$$

In [14] (which might be considered as a first part of the work published in the present article) the author replaced this ellipsoid by an optimal double paraboloid, thereby improving (3.1) to

$$\theta_3 = \Delta(K_4) \geq 1.159^3\,\tfrac{1}{2}\left(1+\sqrt{2}\right) = 1.879\ldots \tag{3.2}$$

Using (1.5) and Theorem 1 with $s = p = 4$, one obtains

$$\theta_4 = \Delta(K_5) \geq \Delta(K^*_{4,4})^{5/3} \geq \Delta(K^*_{3,4})^{20/9}\,\Delta(K^{**}_{4,4})^{5/9}\,.$$

One then can inscribe a 4-dimensional ellipsoid into $K^{**}_{4,4}$ and a double paraboloid into $\Delta(K^*_{3,4})$. This was also done in [14], infering the estimate

$$\theta_4 = \Delta(K_5) \geq 1.1621^{20/9}\left(\frac{\sqrt{80}}{6\sqrt[5]{12}}\right)^{5/9} = 1.3225\ldots \tag{3.3}$$

We can now carry on this approach further: The salient point is that, starting from (1.5) and applying Theorem 1 repeatedly, one can always derive an inequality for $\Delta(K_{s+1})$ which involves a sequence of star bodies with dimensions decreasing down to 3.[(4)] For $s = 5$, we thus obtain

$$\begin{aligned}\theta_5 = \Delta(K_6) \geq \Delta(K^*_{5,5})^{3/2} &\geq \Delta(K^*_{4,5})^{15/8}\,\Delta(K^{**}_{5,5})^{3/8} \geq \\ &\geq \Delta(K^*_{3,5})^{5/2}\,\Delta(K^{**}_{4,5})^{5/8}\,\Delta(K^{**}_{5,5})^{3/8}\,.\end{aligned} \tag{3.4}$$

(4) Of course it is possible to go one step further and arrive at a planar domain. But numerical details strongly suggest that, for $s \geq 4$, this would yield a weaker bound.

It remains to estimate the critical determinants of these last three star bodies. For dimension ≥ 4, there is no other possibility in sight than inscribing an optimal ellipsoid and using the critical determinants of the unit spheres which are known up to dimension 8 (see Gruber/Lekkerkerker [6], p. 410). Although this inscribing procedure is straightforward via the mean inequality, it may be convenient to have at hand the following auxiliary result.

Lemma 2. *For integers $p \geq s \geq 2$, let the star body $K^{**}_{s,p}$ be defined as in Theorem 1, and denote by $\mathcal{S}_s$ the unit sphere in $\mathbb{R}^s$. Then it follows that*

$$\Delta(K^{**}_{s,p}) \geq \Delta(\mathcal{S}_s)\, \frac{(p+1)^{s/2}}{(p-s+2)^{1/2}} \left((p-s+2)\lambda_0^s + (s-1)\lambda_0^{-1-p+s}\right)^{-(s-1)/2}$$

where $\lambda_0 = \left(\frac{(p-s+1)(s-1)}{(p-s+2)s}\right)^{1/(p+1)}$.

Proof. With a parameter $\lambda > 0$ at our disposition, the mean inequality yields

$$\begin{aligned}
&(x_1^2 + \ldots + x_s^2)^{p-s+2}(x_2^2 + \ldots + x_s^2)^{s-1} = \\
&= \left(\lambda^{s-1}(x_1^2 + \ldots + x_s^2)\right)^{p-s+2} \left(\lambda^{-(p-s+2)}(x_2^2 + \ldots + x_s^2)\right)^{s-1} \leq \\
&\leq \left(\tfrac{\lambda^{s-1}(p-s+2)x_1^2 + ((p-s+2)\lambda^{s-1} + (s-1)\lambda^{-(p-s+2)})(x_2^2+\ldots+x_s^2)}{p+1}\right)^{p+1} .
\end{aligned}$$

Thus the ellipsoid

$$\mathcal{E}(\lambda): \quad \tfrac{\lambda^{s-1}(p-s+2)}{p+1} x_1^2 + \tfrac{(p-s+2)\lambda^{s-1} + (s-1)\lambda^{-(p-s+2)}}{p+1}(x_2^2 + \ldots + x_s^2) \leq 1$$

is contained in $K^{**}_{s,p}$. The linear map

$$T(\lambda): \begin{cases} x_1 \to \left(\tfrac{\lambda^{s-1}(p-s+2)}{p+1}\right)^{1/2} x_1\,, \\ x_j \to \left(\tfrac{(p-s+2)\lambda^{s-1} + (s-1)\lambda^{-(p-s+2)}}{p+1}\right)^{1/2} x_j \quad (j = 2, \ldots, s) \end{cases}$$

transforms $\mathcal{E}(\lambda)$ into $\mathcal{S}_s$. Its determinant is

$$\begin{aligned}
\det T(\lambda) &= \left(\tfrac{\lambda^{s-1}(p-s+2)((p-s+2)\lambda^{s-1} + (s-1)\lambda^{-(p-s+2)})^{s-1}}{(p+1)^s}\right)^{1/2} \\
&= \frac{(p-s+2)^{1/2}}{(p+1)^{s/2}} \left((p-s+2)\lambda^s + (s-1)\lambda^{-1-p+s}\right)^{(s-1)/2} .
\end{aligned}$$

Of course,

$$\Delta(K^{**}_{s,p}) \geq \Delta(\mathcal{E}(\lambda)) = (\det T(\lambda))^{-1}\Delta(\mathcal{S}_s)\,,$$

and the right-hand side becomes maximal if $(p-s+2)\lambda^s + (s-1)\lambda^{-1-p+s}$ attains its minimum, which obviously happens for $\lambda = \lambda_0$. This establishes the Lemma.

Applying Lemma 2 to (3.4) (along with the known values $\Delta(\mathcal{S}_4)=\frac{1}{2}$, $\Delta(\mathcal{S}_5)=\frac{\sqrt{2}}{4}$), we arrive at

$$\theta_5=\Delta(K_6)\ge\frac{5^{5/8}}{2^{11/16}\,3^{3/4}}\,\Delta(K^*_{3,5})^{5/2}\,. \tag{3.5}$$

To estimate $\Delta(K^*_{3,5})$ we make use of the main result of the author's previous article [14]: The critical determinant of the double paraboloid

$$\mathcal{P}\;:\quad x^2+y^2+|z|\le 1$$

is given by

$$\Delta(\mathcal{P})=\frac{1}{2}\,. \tag{3.6}$$

Since $K^*_{3,5}$ is a body of revolution with respect to the x_1-axes, we inscribe a double paraboloid (with a parameter $q>0$ remaining at our disposition)

$$\mathcal{P}_{(q)}:\quad |x_1|+\frac{1}{q}\left(x_2^2+x_3^2\right)\le 1\,.$$

It suffices to discuss the situation in front view, i.e. in (the right half of) an (x,r)-plane, say. We have to determine q such that the parabola

$$x=1-\frac{r^2}{q}$$

touches the curve

$$x^2\left(x^2+r^2\right)^5=1$$

from the left. After eliminating r^2 from these two equations, the point is that the function

$$f(x)=x^2\left(x^2+q(1-x)\right)^5-1$$

has a double root, or, equivalently, that the resultant of $f(x)$ and $f'(x)$ vanishes. In the syntax of MATHEMATICA [18], this reads

```
f[x_] = x^2(x^2 + q(1 - x))^5 - 1;    f1[x_] = D[f[x], x];
rho[q_] = Resultant[f[x], f1[x], x];    NSolve[rho[q] == 0, q]
```

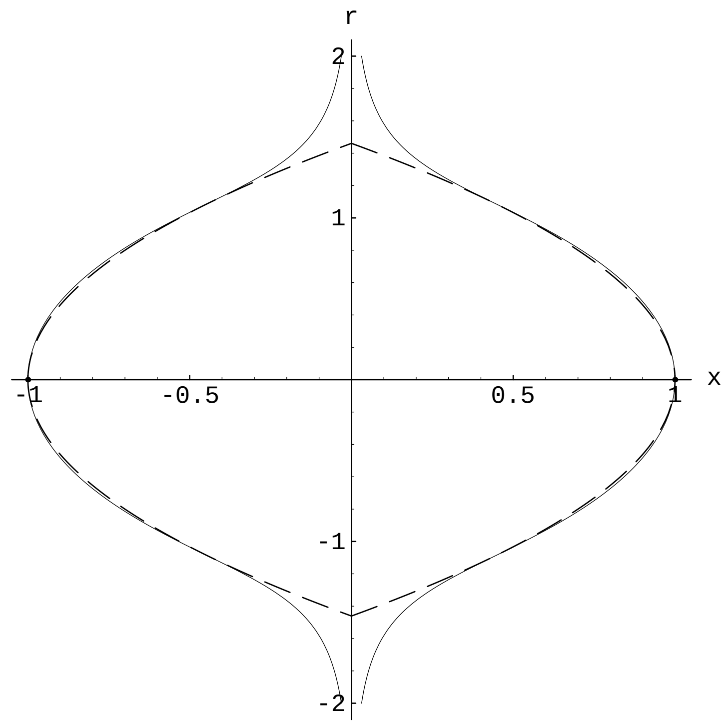
It turns out that the smallest positive real solution is $q=2.1341\ldots$. By (3.6),

$$\Delta(K^*_{3,5})\ge\Delta(\mathcal{P}_{(q)})=\frac{q}{2}\ge 1.06705$$

which together with (3.5) gives

$$\theta_5=\Delta(K_6)\ge 0.876\,.$$

A picture shows how marvelously the paraboloid fits into the (non-convex) body $K^*_{3,5}$.



The bodies $K^*_{3,5}$ and $\mathcal{P}_{(q)}$ $(q = 2.1341\ldots)$ in front view.

We can state the result obtained as follows:

Theorem 2. *Let* θ_5, *the Diophantine approximation constant in* $\mathbb{R}^5$ *with respect to the Euclidean norm, be defined as the supremum of all reals* c *with the property that for any real but not all rational* 5*-tupel* $(x_1, \ldots, x_5)$ *there exist infinitely many* $(p_1, \ldots, p_5, q) \in \mathbb{Z}^5 \times \mathbb{N}^*$ *with* $(x_1 - \frac{p_1}{q})^2 + \ldots + (x_5 - \frac{p_5}{q})^2 < c^{-2/5} q^{-12/5}$. *Then the estimate*

$$\theta_5 \geq 0.876\,.$$

holds true.

Remarks. 1. It should be pointed out that one salient point in the derivation of all these new bounds is to have at hand the result (3.6) for the critical determinant of the double paraboloid which turns out to be a particular useful body in this context. A result like our Theorem 1 certainly might have been in reach of earlier authors but it would have been lacking immediate applications without (3.6).

2. Numerically, the estimate of Theorem 2 clearly supersedes the bound contained in (1.3) which is $\theta_5 \geq 0.836\ldots$. However, the very same approach (at the present state-of-art) fails to improve upon (1.3) for $s \geq 6$. Perhaps further work will give some possibility to estimate the critical determinants of higher dimensional bodies better than by inscribing ellipsoids and thereby yield new bounds for some more of these approximation constants.

4. Upper bounds for the θ_s. Here we can employ well- established arguments (involving a bit of algebraic number theory), following the example of section 6 in Armitage [1]. For arbitrary $s \geq 2$, let throughout $r = r(s) = \begin{cases} 2 & \text{if } s \text{ is odd,} \\ 1 & \text{if } s \text{ is even.} \end{cases}$
We consider an auxiliary $(s+1)$-dimensional star body K_{s+1}^{+} defined as follows:

$$K_{s+1}^{+}: \quad \begin{cases} |x_0 x_1| \, (x_2^2 + x_3^2) \dots (x_{s-1}^2 + x_s^2) \leq 1 & \text{if } s \text{ is odd,} \\ |x_0| \, (x_1^2 + x_2^2) \dots (x_{s-1}^2 + x_s^2) \leq 1 & \text{if } s \text{ is even.} \end{cases}$$

Let $\mathbb{F}$ be an algebraic number field of degree $s+1$ with exactly r real conjugates and denote its discriminant by D. Then it readily follows that

$$\Delta(K_{s+1}^{+}) \leq 2^{-(s+1-r)/2} \sqrt{|D|}\,, \tag{4.1}$$

since $(\xi^{(0)}, \dots, \xi^{(r-1)}, \Re\xi^{(r)}, \Im\xi^{(r)}, \dots, \Re\xi^{(s)}, \Im\xi^{(s)})$ (with ξ ranging over all algebraic integers of $\mathbb{F}$, the superscript meaning conjugates) is an admissible lattice for K_{s+1}^{+} with lattice constant $2^{-(s+1-r)/2}\sqrt{|D|}$ (see Gruber/Lekkerkerker [6], p. 30, Theorem 2). We use the mean inequality in the form

$$\left(x_1^2 (x_2^2 + x_3^2)^2 \dots (x_{s-1}^2 + x_s^2)^2\right)^{1/s} \leq \frac{1}{s} \left(x_1^2 + 2(x_2^2 + \ldots + x_s^2)\right)$$

for s odd, and in the shape

$$\left((x_1^2 + x_2^2) \dots (x_{s-1}^2 + x_s^2)\right)^{2/s} \leq \frac{2}{s} (x_1^2 + \ldots + x_s^2)$$

for s even. This shows that, in both cases, $\tau^{-1}(K_{s+1}) \subseteq K_{s+1}^{+}$ where τ is a suitable linear transformation in $\mathbb{R}^{s+1}$, namely

$$\tau: \quad \begin{cases} x_0 \to x_0, \ x_1 \to \frac{x_1}{\sqrt{s}}, \ x_j \to \frac{\sqrt{2}}{\sqrt{s}} x_j \ (j = 2, \dots, s) & \text{for } s \text{ odd,} \\ x_0 \to x_0, \ x_j \to \frac{\sqrt{2}}{\sqrt{s}} x_j \ (j = 1, \dots, s) & \text{for } s \text{ even.} \end{cases}$$

Clearly $\det \tau = 2^{(s+1-r)/2} s^{-s/2}$ in both cases, hence, by (4.1),

$$\Delta(K_{s+1}) \leq |\det \tau| \, \Delta(K_{s+1}^{+}) \leq s^{-s/2} \sqrt{|D|}\,.$$

We summarize this result as follows.

Theorem 3. *For any $s \geq 2$, let D denote the discriminant of some algebraic number field of degree $s+1$ with exactly r real conjugates ($r = r(s)$ as above), then it follows that*

$$\theta_s = \Delta(K_{s+1}) \leq s^{-s/2} \sqrt{|D|}\,.$$

Using number fields satisfying the constraints stated with minimal absolute discriminant (see, e.g., the Appendix of the textbook by POHST & ZASSENHAUS [15] (5)), one readily obtains

$$\theta_1 = \Delta(K_2) \le \sqrt{5}$$
$$\theta_2 = \Delta(K_3) \le \tfrac{1}{2}\sqrt{23} \quad \text{(These two bounds are sharp! cf. section 1)}$$
$$\theta_3 = \Delta(K_4) \le 3^{-3/2}\sqrt{275} = 3.1914\ldots, \quad \text{(cf. ARMITAGE [1])}$$
$$\theta_4 = \Delta(K_5) \le 4^{-2}\sqrt{1609} = 2.5070\ldots,$$
$$\theta_5 = \Delta(K_6) \le 5^{-5/2}\sqrt{28037} = 2.9953\ldots.$$

References

[1] ARMITAGE, J.V.: On a method of Mordell in the geometry of numbers. Mathematika **2** (1955), 132-140.

[2] BLICHFELDT, H.: A new principle in the geometry of numbers, with some applications. Trans. Am. Math. Soc. **15** (1914), 227-235.

[3] CASSELS, J.W.S.: Simultaneous Diophantine approximation. J. London Math. Soc. **30** (1955), 119-121.

[4] DAVENPORT, H.: On a theorem of Furtwängler. J. London Math. Soc. **30** (1955), 185-195.

[5] DAVENPORT, H., and MAHLER, K.: Simultaneous Diophantine approximation. Duke Math. J. **13** (1946), 105-111.

[6] GRUBER, P.M., and LEKKERKERKER, C.G.: Geometry of numbers. North Holland, Amsterdam-New York-Oxford-Tokyo, 1987.

[7] LEKKERKERKER, C.G.: Eine Mordell'sche Methode in der Geometrie der Zahlen. J. reine angew. Math. **206** (1961), 20-25.

[8] MACK, J.M.: Simultaneous Diophantine approximation. J. Austral. Math. Soc. (Ser. A) **24** (1977), 266-285.

[9] MORDELL, L.J.: The product of three homogeneous linear ternary forms. J. London Math. Soc. **17** (1942), 107-115.

[10] MORDELL, L.J.: Observation on the minimum of a positive quadratic form in eight variables. J. London Math. Soc. **19** (1944), 3-6.

[11] MORDELL, L.J.: On the minimum of a ternary cubic form. J. London Math. Soc. **19** (1944), 6-12.

[12] NOWAK, W.G.: A note on simultaneous Diophantine approximation. Manuscr. math. **36** (1981), 33-46.

[13] NOWAK, W.G.: On simultaneous Diophantine approximation. Rend. Circ. Math. Palermo, II. Ser., **33** (1984), 456-460.

[14] NOWAK, W.G.: The critical determinant of the double paraboloid and Diophantine approximation in $\mathbb{R}^3$ and $\mathbb{R}^4$. Math. Pannonica **10** (1999), 111-122.

[15] POHST, M., and ZASSENHAUS, H.: Algorithmic algebraic number theory. Encyclopedia of mathematics and its applications. Cambridge Univ. Press, Cambridge 1989.

(5) The author is indebted to Professor HALTER-KOCH for providing this important reference.

[16] Prasad, A.V.: Simultaneous Diophantine approximation. Proc. Indian Acad. Sci., A **31** (1950), 1-15.

[17] Spohn, W.G.: Blichfeldt's theorem and simultaneous Diophantine approximation. Amer. J. Math. **90** (1968), 885-894.

[18] Wolfram Research, Inc.: Mathematica, Version 2.2. Wolfram Research, Inc., Champaign, 1993.

Werner Georg Nowak
Institut für Mathematik
Universität für Bodenkultur
A-1180 Wien, AUSTRIA

E-mail: `nowak@mail.boku.ac.at`

http://www.boku.ac.at/math/nth.html