

# The critical determinant of the double paraboloid and Diophantine approximation in $\mathbb{R}^3$ and $\mathbb{R}^4$

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*Abstract.* In this note the critical determinant of the double paraboloid  $x^2 + y^2 + |z| \leq 1$  is evaluated and the critical lattices are determined. As an application, new bounds for the simultaneous Diophantine approximation constants in  $\mathbb{R}^3$  and  $\mathbb{R}^4$  (with respect to the Euclidean norm) are obtained.

*AMS-Classification:* 11H06, 11J13

**1. Introduction.** Let  $K$  be a body in  $\mathbb{R}^s$ ,  $s \geq 2$ , starlike with respect to the origin  $\mathbf{o}$ , and  $\Lambda = \mathbf{A}\mathbb{Z}^s$  an  $s$ -dimensional lattice where  $\mathbf{A}$  is a real non-singular  $(s \times s)$ -matrix.  $\Lambda$  is called *admissible* for  $K$  if the only lattice point of  $\Lambda$  contained in the interior of  $K$  is  $\mathbf{o}$ . Further, the *critical determinant*  $\Delta(K)$  of the body  $K$  is defined as the infimum of all *lattice constants*  $d(\Lambda) = |\det \mathbf{A}|$  where  $\Lambda$  ranges over all lattices admissible for  $K$ . Finally, a lattice  $\Lambda$  is called *critical* for  $K$ , if it is admissible and satisfies  $d(\Lambda) = \Delta(K)$ . (Cf. throughout GRUBER/LEKKERKERKER [8].)

The evaluation or estimation of critical determinants for various special bodies  $K$  was a central problem in the classic age of the Geometry of Numbers until the mid of this century. As one motivation, among lots of other aspects, one might mention the following connection with simultaneous Diophantine approximation: Let  $\|\cdot\|_\nu$  denote any  $\nu$ -norm in  $\mathbb{R}^s$ , then the  $s$ -dimensional simultaneous approximation constant (with respect to  $\|\cdot\|_\nu$ )  $\theta_{s,\nu}$  is defined as the supremum of all reals  $c$  with the property that, for any  $s$ -tuple  $\mathbf{x} \in \mathbb{R}^s - \mathbb{Q}^s$ , there exist infinitely many  $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{N}^*$  satisfying

$$\left\| \mathbf{x} - \frac{1}{q} \mathbf{p} \right\|_\nu < \frac{1}{c^{1/s} q^{1+1/s}}. \quad (1.1)$$

By Hurwitz' classic theorem,  $\theta_{1,\nu} = \sqrt{5}$  (independently of  $\nu$  of course). According to a deep result of DAVENPORT and MAHLER [7],  $\theta_{2,2} = \frac{1}{2}\sqrt{23}$ . For all other  $(s, \nu)$ , the exact values of  $\theta_{s,\nu}$  are unknown with only more or less precise bounds available. References may be found in the author's previous articles [10], [11].

By a celebrated theorem of DAVENPORT [6] (see also [8], p. 480, Theorem 4),  $\theta_{s,\nu}$  is equal to the critical determinant of the  $(s+1)$ -dimensional star body

$$K_{s+1} : |x_0| (\|(x_1, \dots, x_s)\|_\nu)^s \leq 1. \quad (1.2)$$

For the special case of a *convex,  $\mathbf{o}$ -symmetric* body  $K$  in  $\mathbb{R}^3$ , MINKOWSKI [9] already developed a method which at least in principle may be efficient to evaluate the

critical determinant  $\Delta(K)$ : He proved (cf. [8], p. 342, Theorem 3) that there always exists a critical lattice with a basis  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ , such that *either*

(I)  $\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{c}, \pm(\mathbf{a} - \mathbf{b}), \pm(\mathbf{b} - \mathbf{c}), \pm(\mathbf{a} - \mathbf{c})$  lie on the boundary  $\partial K$  of  $K$ , and  $\pm(\mathbf{a} + \mathbf{b} - \mathbf{c}), \pm(\mathbf{a} + \mathbf{c} - \mathbf{b}), \pm(\mathbf{b} + \mathbf{c} - \mathbf{a})$  lie outside  $K$ , *or*,

(II)  $\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{c}, \pm(\mathbf{a} + \mathbf{b}), \pm(\mathbf{b} + \mathbf{c}), \pm(\mathbf{a} + \mathbf{c})$  lie on  $\partial K$  and  $\pm(\mathbf{a} + \mathbf{b} + \mathbf{c})$  do not lie in the interior of  $K$ .

(In the case of strict convexity, each critical lattice is subject to one of these conditions.)

Equipped with this tool, one can attack the determination of  $\Delta(K)$  as a minimum problem with constraints, facing, however, overwhelming technical difficulties in most special cases. This task was carried out by MINKOWSKI himself for the octahedron

$$|x| + |y| + |z| \leq 1$$

( $\Delta(K) = \frac{19}{108}$ ), by OLLERENSHAW [12] for the unit sphere ( $\Delta(K) = \frac{1}{\sqrt{2}}$ ), by WHITWORTH [13], [14] for the double cone

$$\sqrt{x^2 + y^2} + |z| \leq 1$$

( $\Delta(K) = \frac{1}{8}\sqrt{6}$ ) and another body which generalizes both the cube and the octahedron; further, by CHALK [3] for the "frustrum of a sphere"

$$x^2 + y^2 + z^2 \leq 1, \quad |z| \leq \alpha < 1,$$

and by AP SIMON [1] for an off-centered sphere.

Concerning *non-convex* three-dimensional star bodies we mention just the results of DAVENPORT [4], [5]: For the bodies

$$K_1 : |xyz| \leq 1, \quad K_2 : (x^2 + y^2) |z| \leq 1,$$

one has

$$\Delta(K_1) = 7, \quad \text{and} \quad \Delta(K_2) = \frac{1}{2}\sqrt{23}.$$

We conclude this section with an appeal to the work of WOLFF [15] who established a (rather elaborate) generalization of Minkowski's conditions (I), (II) to the four-dimensional case.

**2. Statement of results. 2.1. The paraboloid.** The objective of the present article is to take up again the program initiated by Minkowski and to carry out the necessary hard analysis for one more convex body – encouraged, as we admit frankly, by the idea that nowadays software packages for symbolic computations should prove helpful in overcoming the difficulties arising in the details of the arguments. In particular, our aim is the determination of the critical determinant and critical lattices of the convex body  $\mathcal{P}$  which is bounded by the two paraboloid surfaces  $\mathcal{P}_+, \mathcal{P}_-$  defined as

$$\mathcal{P}_\pm : z = \pm(1 - x^2 - y^2), \quad x^2 + y^2 \leq 1. \quad (2.1)$$

We are able to prove the following.

**Theorem 1.** *The critical determinant of  $\mathcal{P}$  is given by*

$$\Delta(\mathcal{P}) = \frac{1}{2},$$

the critical lattices are those (up to rotations around the  $z$ -axes) generated by the vectors

$$\mathbf{a} = \begin{pmatrix} u - \frac{1}{2u} \\ \pm \left(u - \frac{1}{2u}\right) \\ 3 - 2u^2 - \frac{1}{2u^2} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} u - \frac{1}{2u} \\ \mp \frac{1}{2u} \\ 2 - u^2 - \frac{1}{2u^2} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} u \\ 0 \\ 1 - u^2 \end{pmatrix},$$

where  $u$  is any real number satisfying  $\frac{1}{\sqrt{2}} \leq |u| \leq 1$ .

**2.2. The Euclidean approximation constants in  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .** While, as we said earlier, the Diophantine approximation constants  $\theta_{1,2} = \sqrt{5}$  and  $\theta_{2,2} = \frac{1}{2}\sqrt{23}$  are known, the hitherto sharpest bounds for  $\theta_{3,2}$  are due to ARMITAGE [2] and read

$$3.1914\dots = \sqrt{\frac{275}{27}} \geq \theta_{3,2} \geq 1.159^3 \frac{1}{2} 3^{3/4} = 1.774\dots \quad (2.2)$$

Starting from the fact that  $\theta_{3,2} = \Delta(K_4)$  where  $K_4$  is the 4- dimensional star body

$$K_4 : \quad |x_0| (x_1^2 + x_2^2 + x_3^2)^{3/2} \leq 1$$

(cf. (1.2)), ARMITAGE applied a method of MORDELL to reduce the problem to lower dimensions; he showed that

$$\Delta(K_4) \geq (\Delta(K_2^*))^3 \Delta(K_3^*) \quad (2.3)$$

where  $K_2^*$  is the planar domain

$$K_2^* : \quad |x| (x^2 + y^2)^{3/2} \leq 1$$

and  $K_3^*$  is the three-dimensional body

$$K_3^* : \quad (x^2 + y^2) (x^2 + y^2 + z^2) \leq 1. \quad (2.4)$$

Quoting the result  $\Delta(K_2^*) \geq 1.159$  from his London Ph.D. dissertation <sup>1)</sup>, he estimated  $\Delta(K_3^*)$  by inscribing an ellipsoid and using  $\Delta(S_3) = \frac{1}{\sqrt{2}}$ ,  $S_3$  the unit sphere in  $\mathbb{R}^3$ .

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<sup>1)</sup> The author is indebted to Professor ARMITAGE for the courtesy of sending him a carbon copy (!) of his 1956 thesis.

We are able to improve upon his lower bound for  $\theta_{3,2}$  by replacing his ellipsoid by a suitable double paraboloid.

Furthermore, we apply his method to deal with  $\theta_{4,2}$  as well.

**Theorem 2.** *Define  $\theta_{s,2}$ , the Euclidean Diophantine approximation constant in  $\mathbb{R}^s$ , as the supremum of all reals  $c$  with the property that for any real but not all rational  $s$ -tupel  $(x_1, \dots, x_s)$  there exist infinitely many  $(p_1, \dots, p_s, q) \in \mathbb{Z}^s \times \mathbb{N}^*$  satisfying*

$$\left(x_1 - \frac{p_1}{q}\right)^2 + \dots + \left(x_s - \frac{p_s}{q}\right)^2 < c^{-2/s} q^{-2-2/s}.$$

Then the estimates

$$\theta_{3,2} \geq 1.159^3 \frac{1}{2} (1 + \sqrt{2}) = 1.879\dots \quad (2.5)$$

and

$$\theta_{4,2} \geq \left(\frac{\sqrt{80}}{6\sqrt[5]{12}}\right)^{5/9} 1.1621^{20/9} = 1.3225\dots \quad (2.6)$$

hold true.

**3. Proof of Theorem 1.** Fortunately for our purpose, WHITWORTH [13] has established some simplifications of Minkowski's conditions (I), (II) which are especially useful in a situation where the boundary of the convex body is divided into an upper and a lower half described by a pair of formally different equations. In our notation, we can state his results as follows: Assuming throughout that a critical lattice  $\Lambda$  is generated by three points  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  which lie on  $\mathcal{P}_+$ , it suffices to consider the following 5 cases:

(I.1)  $\mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{c}, \mathbf{b} - \mathbf{c}$  lie on  $\mathcal{P}_+$ , and  $\mathbf{a} + \mathbf{b} - \mathbf{c}, \mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{a} + \mathbf{c} - \mathbf{b}$  are not in the interior of  $\mathcal{P}$ .

(II.0)  $\mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}$  lie on  $\mathcal{P}_+$ , and  $\mathbf{a} + \mathbf{b} + \mathbf{c}$  is not in the interior of  $\mathcal{P}$ .

(II.1)  $\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{c}, \mathbf{b} - \mathbf{c}$  lie on  $\mathcal{P}_+$ , and  $\mathbf{a} + \mathbf{b} - \mathbf{c}$  is not in the interior of  $\mathcal{P}$ .

(II.2)  $\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{c}, \mathbf{c} - \mathbf{b}$  lie on  $\mathcal{P}_+$ , and  $\mathbf{a} + \mathbf{b} - \mathbf{c}$  is not in the interior of  $\mathcal{P}$ .

(II.3)  $\mathbf{a} + \mathbf{b}, \mathbf{c} - \mathbf{a}, \mathbf{c} - \mathbf{b}$  lie on  $\mathcal{P}_+$ , and  $\mathbf{a} + \mathbf{b} - \mathbf{c}$  is not in the interior of  $\mathcal{P}$ .

It will in fact turn out that the critical lattices described in Theorem 1 arise all from case (I.1), the other cases (II.0) – (II.3) being essentially void. In order to state this later in full precision, it will be necessary to have at hand the description of 4 particular lattices  $\Lambda_*$ , say: Let us put, in the critical lattices of Theorem 1,  $u = \frac{\varepsilon}{\sqrt{2}}$ ,  $\varepsilon \in \{1, -1\}$ . Choosing as a new basis  $\mathbf{a}^* = \mathbf{a} - \mathbf{b}, \mathbf{b}^* = \mathbf{b}, \mathbf{c}^* = \mathbf{c}$ , we readily obtain the lattices

$$\Lambda_* : \quad \mathbf{a}^* = \begin{pmatrix} 0 \\ \frac{\varepsilon_1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{b}^* = \begin{pmatrix} 0 \\ -\frac{\varepsilon_1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{c}^* = \begin{pmatrix} \frac{\varepsilon_2}{\sqrt{2}} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (\varepsilon_1, \varepsilon_2 \in \{1, -1\}). \quad (3.1)$$

We are now ready to carry out the details of the analysis. We assume that  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{P}_+$  form a basis of some critical lattice  $\Lambda$  of  $\mathcal{P}$ , i.e. (w.l.o.g. by rotational symmetry),

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ 1 - a_1^2 - a_2^2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ 1 - b_1^2 - b_2^2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ 0 \\ 1 - c_1^2 \end{pmatrix},$$

the third components being  $\geq 0$  throughout.

**Case (I.1):** The conditions  $\mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{c}, \mathbf{b} - \mathbf{c} \in \mathcal{P}_+$  yield, after obvious simplifications, the equations

$$\begin{aligned} 1 + 2(a_1b_1 + a_2b_2) &= 2(b_1^2 + b_2^2), \\ 1 + 2a_1c_1 &= 2c_1^2, \\ 1 + 2b_1c_1 &= 2c_1^2. \end{aligned}$$

Solving these for  $a_1, a_2, b_1$  in terms of  $b_2, c_1$  gives

$$a_1 = c_1 - \frac{1}{2c_1}, \quad a_2 = b_2 - \frac{1}{2b_2}, \quad b_1 = c_1 - \frac{1}{2c_1}. \quad (3.2)$$

Using this to express  $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$  in terms of  $b_2, c_1$  (supported, e.g., by the software package MATHEMATICA [16]), things miraculously simplify to

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -\frac{1}{8b_2c_1} - \frac{b_2c_1}{2}.$$

Therefore,  $|\det(\mathbf{a}, \mathbf{b}, \mathbf{c})|$  attains its minimal value  $\frac{1}{2}$  for  $b_2c_1 = \pm\frac{1}{2}$ . Since  $\Lambda$  should be critical we conclude that  $b_2 = \pm\frac{1}{2c_1}$ . Inserting this into (3.2) (and writing  $u$  instead of  $c_1$  for short), we obtain the lattice basis  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  as stated in Theorem 1, apart from the restriction on  $u$ . To derive the latter, we use that the third components of  $\mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{c}, \mathbf{b} - \mathbf{c}$  must be  $\geq 0$  and that  $\mathbf{a} + \mathbf{b} - \mathbf{c}, \mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{a} + \mathbf{c} - \mathbf{b}$  are not in the interior of  $\mathcal{P}$ . This leads to  $\frac{1}{\sqrt{2}} \leq |u| \leq 1$  as asserted, thereby completing the proof of Theorem 1, as far as case (I.1) is concerned.

**Case (II.0):** The conditions  $\mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c} \in \mathcal{P}_+$  give

$$\begin{aligned} 2(a_1b_1 + a_2b_2) &= -1, \\ 2a_1c_1 &= -1, \\ 2b_1c_1 &= -1. \end{aligned}$$

Solving for  $a_1, a_2, b_1$ , we obtain

$$a_1 = -\frac{1}{2c_1}, \quad a_2 = -\frac{1}{2b_2} - \frac{1}{4b_2c_1^2}, \quad b_1 = -\frac{1}{2c_1}.$$

Inserting this yields  $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$  identically, hence a contradiction.

**Case (II.1):** The conditions  $\mathbf{a} + \mathbf{b}$ ,  $\mathbf{a} - \mathbf{c}$ ,  $\mathbf{b} - \mathbf{c} \in \mathcal{P}_+$  imply

$$2(a_1b_1 + a_2b_2) = -1, \quad (3.3)$$

$$1 + 2a_1c_1 = 2c_1^2, \quad (3.4)$$

$$1 + 2b_1c_1 = 2c_1^2, \quad (3.5)$$

$$\max(a_1^2 + a_2^2, b_1^2 + b_2^2) \leq c_1^2 \leq 1. \quad (3.6)$$

We claim that then necessarily  $\mathbf{a} + \mathbf{b} - \mathbf{c}$  is in the interior of  $\mathcal{P}$ , unless  $\Lambda$  is just of the form  $\Lambda_*$  described in (3.1). This will be clear if we verify that

$$0 \leq 1 - (a_1^2 + a_2^2 + b_1^2 + b_2^2) + c_1^2 < 1 - (a_1 + b_1 - c_1)^2 - (a_2 + b_2)^2. \quad (3.7)$$

The first inequality is evident by (3.6). The second one is equivalent to

$$2c_1^2 + \underbrace{2(a_1b_1 + a_2b_2)}_{(3.3)} - \underbrace{2(a_1c_1 + b_1c_1)}_{(3.4), (3.5)} < 0 \iff -2c_1^2 + 1 < 0 \iff c_1^2 > \frac{1}{2} \quad (*)$$

which remains to be proved.

On the other hand, by (3.3), Cauchy's inequality, and (3.6),

$$\frac{1}{2} = |a_1b_1 + a_2b_2| \leq \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2} \leq c_1^2,$$

which establishes (\*), except for the case of equality throughout. But in this case,

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 = \frac{1}{2}. \quad (3.8)$$

By (3.4) and (3.5), we conclude that  $a_1 = b_1 = 0$ , hence, from (3.3) and (3.8),

$$a_2^2 = b_2^2 = -a_2b_2 = \frac{1}{2}.$$

This shows that  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  generate one of the lattices  $\Lambda_*$  described in (3.1).

**Case (II.2):** The conditions  $\mathbf{a} + \mathbf{b}$ ,  $\mathbf{a} - \mathbf{c}$ ,  $\mathbf{c} - \mathbf{b} \in \mathcal{P}_+$  imply

$$2(a_1b_1 + a_2b_2) = -1, \quad (3.9)$$

$$1 + 2a_1c_1 = 2c_1^2, \quad (3.10)$$

$$1 + 2b_1c_1 = 2(b_1^2 + b_2^2), \quad (3.11)$$

$$a_1^2 + a_2^2 \leq c_1^2 \leq b_1^2 + b_2^2 \leq 1. \quad (3.12)$$

We again claim (3.7) (i.e.:  $\mathbf{a} + \mathbf{b} - \mathbf{c}$  is in the interior of  $\mathcal{P}$ ), unless  $\Lambda$  is of the form  $\Lambda_*$ . The first inequality is again clear from (3.12). The second one is equivalent to

$$\begin{aligned} 2c_1^2 + \underbrace{2(a_1b_1 + a_2b_2)}_{(3.9)} - \underbrace{2(a_1c_1 + b_1c_1)}_{(3.10), (3.11)} < 0 &\iff -2(b_1^2 + b_2^2) + 1 < 0 \\ &\iff b_1^2 + b_2^2 > \frac{1}{2} \quad (**) \end{aligned}$$

which remains to be proved. Again, from (3.9) and (3.12),

$$\frac{1}{2} = |a_1 b_1 + a_2 b_2| \leq \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2} \leq b_1^2 + b_2^2,$$

which establishes (\*\*), except for the case of equality throughout. But in this case,

$$a_1^2 + a_2^2 = c_1^2 = b_1^2 + b_2^2 = \frac{1}{2}.$$

Hence, in view of (3.10) and (3.11),  $a_1 = b_1 = 0$  and thus, by (3.9),

$$a_2^2 = b_2^2 = -a_2 b_2 = \frac{1}{2}, \quad (3.13)$$

which completes the argument as before.

**Case (II.2):** We employ the same reasoning as before, with a bit more of technical complications. The conditions  $\mathbf{a} + \mathbf{b}, \mathbf{c} - \mathbf{a}, \mathbf{c} - \mathbf{b} \in \mathcal{P}_+$  imply

$$2(a_1 b_1 + a_2 b_2) = -1, \quad (3.14)$$

$$1 + 2a_1 c_1 = 2(a_1^2 + a_2^2), \quad (3.15)$$

$$1 + 2b_1 c_1 = 2(b_1^2 + b_2^2), \quad (3.16)$$

$$c_1^2 \leq \min(a_1^2 + a_2^2, b_1^2 + b_2^2) \leq \max(a_1^2 + a_2^2, b_1^2 + b_2^2) \leq 1. \quad (3.17)$$

We claim again that

$$0 \leq \underbrace{1 - (a_1^2 + a_2^2 + b_1^2 + b_2^2) + c_1^2}_L < 1 - (a_1 + b_1 - c_1)^2 - (a_2 + b_2)^2,$$

unless  $\Lambda$  is of the form  $\Lambda_*$ . Now

$$\begin{aligned} (a_1 + b_1 - c_1)^2 + (a_2 + b_2)^2 &= a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + \underbrace{2(a_1 b_1 + a_2 b_2)}_{(3.14)} - \underbrace{2(a_1 c_1 + b_1 c_1)}_{(3.15), (3.16)} = \\ &= 1 - (a_1^2 + a_2^2 + b_1^2 + b_2^2) + c_1^2 = L, \end{aligned}$$

hence  $L \geq 0$  and it remains to show that

$$L < 1 - L \quad \iff \quad L < \frac{1}{2}.$$

In view of (3.17),

$$L \leq 1 - \frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2). \quad (3.18)$$

Again by classic inequalities and (3.14),

$$\frac{1}{2} = |a_1 b_1 + a_2 b_2| \leq \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2} \leq \frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2),$$

which together with (3.18) establishes  $L < \frac{1}{2}$ , except for the case of equality throughout. In the latter, we have again

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 = \frac{1}{2},$$

hence, from (3.15) and (3.16),  $a_1 = b_1 = 0$ , and (3.13) follows as before, which leads again to a critical lattice of the form  $\Lambda_*$ .

This completes the proof of Theorem 1.

**4. Proof of Theorem 2.** Using the results of ARMITAGE reported in section 2.2, we can start from

$$\theta_{3,2} \geq 1.159^3 \Delta(K_3^*)$$

where  $K_3^*$  is the three-dimensional body given by (2.4). We simply inscribe into  $K_3^*$  a double paraboloid  $\mathcal{P}^{(\alpha)}$  bounded by two paraboloid surfaces  $\mathcal{P}_+^{(\alpha)}, \mathcal{P}_-^{(\alpha)}$  defined as

$$\mathcal{P}_\pm^{(\alpha)} : z = \pm \alpha (1 - x^2 - y^2), \quad x^2 + y^2 \leq 1,$$

where  $\alpha = 1 + \sqrt{2}$ . Since

$$\Delta(K_3^*) \geq \Delta(\mathcal{P}^{(\alpha)}) = \alpha \Delta(\mathcal{P}) = \frac{1}{2}(1 + \sqrt{2}),$$

assertion (2.5) of Theorem 2 is immediate.

It remains, however, to justify the choice of  $\alpha$ : By rotational symmetry around the  $z$ -axes, a discussion in a  $(r, z)$ -plane suffices. The equations of  $\partial K_3^*$  and  $\mathcal{P}_+^{(\alpha)}$  then read

$$\mathcal{C}_1 : r^2(r^2 + z^2) = 1 \quad \text{and} \quad \mathcal{C}_2 : z = \alpha(1 - r^2),$$

respectively. These curves obviously intersect at  $(\pm 1, 0)$ . Eliminating  $r^2$  (and dividing by  $z$ ), we obtain

$$z^2 - \left( \alpha + \frac{1}{\alpha} \right) z + 2 = 0.$$

If the parabola  $\mathcal{C}_2$  touches the curve  $\mathcal{C}_1$  (from below) this equation must have a double root. The relevant condition gives

$$\alpha^4 - 6\alpha^2 + 1 = 0.$$

This has the positive solutions  $\alpha = \sqrt{2} \pm 1$ ; of these, only  $\alpha = 1 + \sqrt{2}$  leads to real  $(r, z)$ -pairs, namely  $(\pm \sqrt{\sqrt{2} - 1}, \sqrt{2})$ .

We illustrate the matter by the following picture which shows (in front view) the non-convex body  $K_3^*$  together with our inscribed double paraboloid  $\mathcal{P}^{(\alpha)}$  and ARMITAGE's ellipsoid

$$\mathcal{E} : \quad \frac{4}{\sqrt{3}} x^2 + \frac{4}{\sqrt{3}} y^2 + \frac{1}{\sqrt{3}} z^2 \leq 2.$$

The improvement we obtained is strongly suggested by this configuration (though  $\mathcal{E}$  is not actually contained in  $\mathcal{P}^{(\alpha)}$ ).

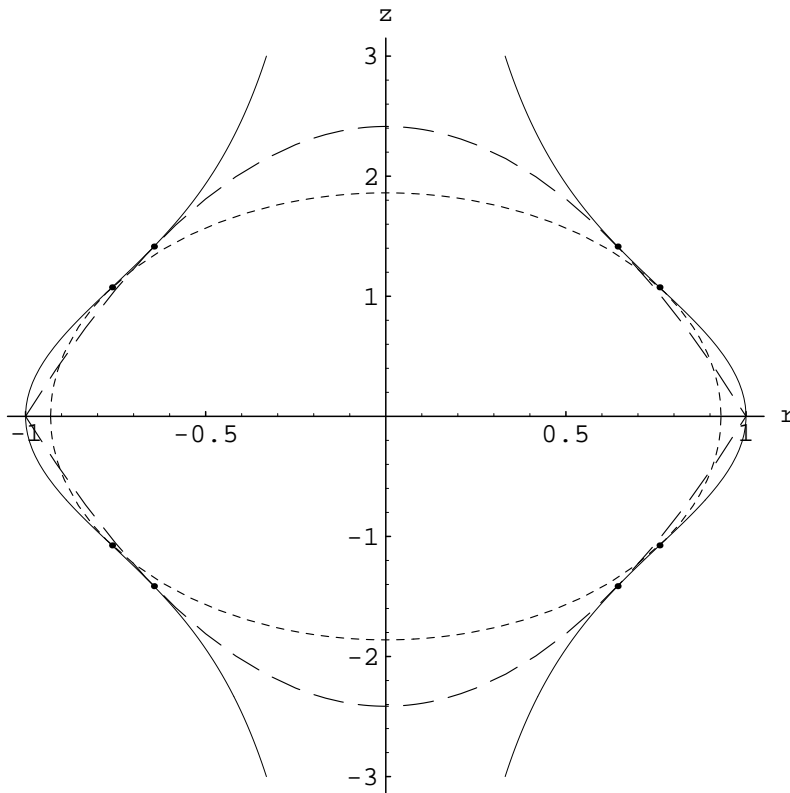


Fig. 1: The bodies  $K_3^*$ ,  $\mathcal{P}^{(\alpha)}$ , and  $\mathcal{E}$  in front view.

We proceed to establish the bound (2.6) for  $\theta_{4,2}$ . Again, this is equal to  $\Delta(K_5)$  where

$$K_5 : |x_0| (x_1^2 + \dots + x_4^2)^2 \leq 1.$$

We remark parenthetically that the obvious idea to inscribe a 5-dimensional ellipsoid yields  $\Delta(K_5) \geq 1.2352\dots$  which was refined only by about  $10^{-3}$  in [10]. By Theorem 2 of ARMITAGE [2],

$$\Delta(K_5) \geq (\Delta(K_4^*))^{5/9} (\Delta(K_3^{**}))^{20/9}, \quad (4.1)$$

where

$$K_4^* : (x_1^2 + \dots + x_4^2)^2 (x_2^2 + x_3^2 + x_4^2)^3 \leq 1,$$

and

$$K_3^{**} : |x| (x^2 + y^2 + z^2)^2 \leq 1.$$

We estimate  $\Delta(K_4^*)$  by inscribing a 4-dimensional ellipsoid. For arbitrary  $\lambda > 0$ , the mean inequality implies

$$\begin{aligned} & \left( (\lambda^3(x_1^2 + \dots + x_4^2))^2 (\lambda^{-2}(x_2^2 + x_3^2 + x_4^2))^3 \right)^{1/5} \leq \\ & \leq \frac{1}{5} \left( 2\lambda^3 x_1^2 + \left( 2\lambda^3 + \frac{3}{\lambda^2} \right) (x_2^2 + x_3^2 + x_4^2) \right). \end{aligned}$$

Therefore, the ellipsoid

$$\mathcal{E}_4(\lambda) : \frac{2}{5}\lambda^3 x_1^2 + \left( \frac{2}{5}\lambda^3 + \frac{3}{5\lambda^2} \right) (x_2^2 + x_3^2 + x_4^2) \leq 1$$

is contained in  $K_4^*$ . Since the critical determinant of the 4-dimensional unit sphere  $S_4$  is equal to  $\frac{1}{2}$  (see WOLFF [15], or [8], p. 410), it follows that

$$\Delta(K_4^*) \geq \Delta(\mathcal{E}_4(\lambda)) = \left( \frac{2}{5}\lambda^3 \left( \frac{2}{5}\lambda^3 + \frac{3}{5\lambda^2} \right)^3 \right)^{-1/2} \Delta(S_4) = \frac{1}{2} \sqrt{\frac{5}{2}} \left( \frac{2}{5}\lambda^4 + \frac{3}{5\lambda} \right)^{-3/2}.$$

This expression attains its maximum value  $\frac{\sqrt{80}}{6\sqrt[5]{12}}$  for  $\lambda = \sqrt[5]{\frac{3}{8}}$ . Consequently,

$$\Delta(K_4^*) \geq \frac{\sqrt{80}}{6\sqrt[5]{12}}. \quad (4.2)$$

It remains to estimate the critical determinant of  $K_3^{**}$ . Since  $K_3^{**}$  is a body of revolution with respect to the  $x$ -axes, we inscribe a double paraboloid (with a parameter  $q > 0$  remaining at our disposition)

$$\mathcal{P}_{(q)} : |x| + \frac{1}{q} (y^2 + z^2) \leq 1.$$

Discussing the situation in (the right half of) the  $(x, r)$ -plane, we obtain curves

$$\mathcal{C}_1 : x(x^2 + r^2)^2 = 1, \quad \mathcal{C}_2 : x = 1 - \frac{r^2}{q}$$

which obviously intersect at  $(x, r) = (1, 0)$ . After eliminating  $r$ , a routine numerical calculation (supported, e.g., again by MATHEMATICA [16]) shows that there is no other point of intersection if we choose<sup>2)</sup>  $q = 2.3242$ . Hence, for this value of  $q$ ,  $\mathcal{P}_{(q)}$  is contained in  $K_3^{**}$ , and it follows that

$$\Delta(K_3^{**}) \geq \Delta(\mathcal{P}_{(q)}) = \frac{q}{2} = 1.1621.$$

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<sup>2)</sup> In fact, the supremum of all  $q$ 's for which this is true is  $2.32422\dots$  and arises as a root of a quintic polynomial.

Together with (4.1) and (4.2) this completes the proof of assertion (2.6) of Theorem 2.

A picture shows how marvelously the paraboloid fits into the (non-convex) body  $K_3^{**}$ .

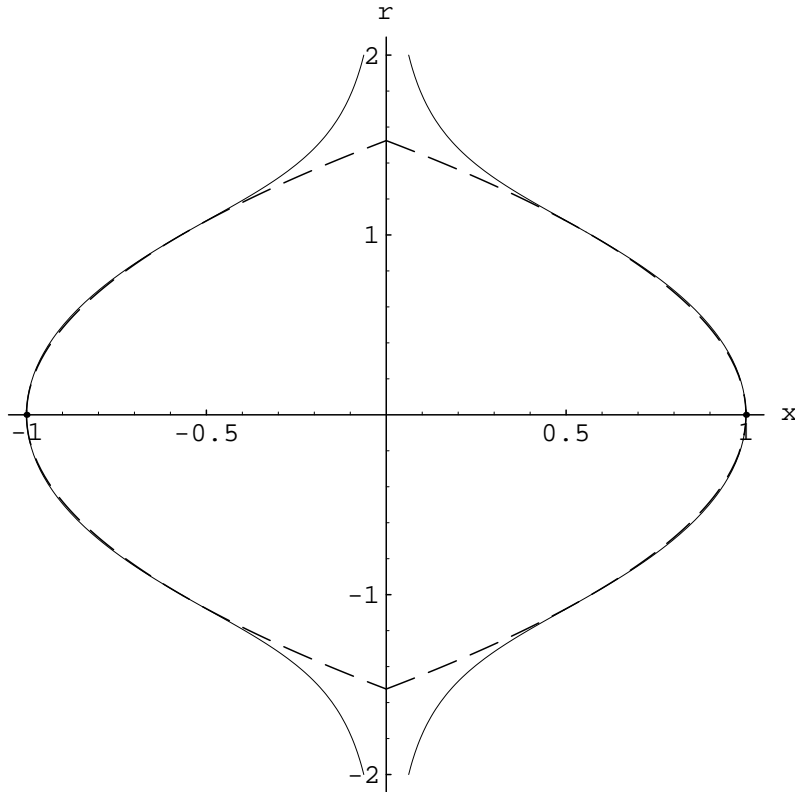


Fig. 2: The bodies  $K_3^{**}$  and  $\mathcal{P}_{(q)}$  ( $q = 2.3242$ ) in front view.

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